Gravity 1 - Recitation 8

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Contents

1 Circular Orbit In Schwarzschild Geometry

1.1 The Basic Equations

The Schwarzschild metric is

$$ds^2 = -\left(1 - \frac{2M}{r}\right) dt^2 + \left(1 - \frac{2M}{r}\right)^{-1} dr^2 + r^2 d\Omega^2 \tag{1}$$

The conserved quantities e and l are

$$e = -\xi \cdot u_p = -g_{tt} u_p^t = \left(1 - \frac{2M}{r}\right) \frac{dt}{d\tau} \tag{2}$$

$$l = \eta \cdot u_p = g_{\phi\phi} u_p^\phi = r^2 sin^2\theta \frac{d\phi}{d\tau} \tag{3}$$

where ξ is the timelike Killing vector $(1, 0, 0, 0)$, η is the Killing vector associated with the azimuthal symmetry $(0, 0, 0, 1)$ and u_p is the particle's 4-velocity.

A stationary observer at radius r has four velocity in coordinate basis

$$u_{ob}^\mu = \left(\left(1 - \frac{2M}{r}\right)^{-\frac{1}{2}}, 0, 0, 0\right) \tag{4}$$

The energy equation for a massive particle is

$$\mathcal{E} = \frac{1}{2} \left(\frac{dr}{d\tau}\right)^2 + V_{eff}(r) \tag{5}$$

where

$$\mathcal{E} = \frac{e^2 - 1}{2} \tag{6}$$

and the effective potential for massive particles is

$$V_{eff}(r) = \frac{1}{2} \left[\left(1 - \frac{2M}{r}\right)\left(1 + \frac{l^2}{r^2}\right) - 1\right] = -\frac{M}{r} + \frac{l^2}{2r^2} - \frac{Ml^2}{r^3} \tag{7}$$

1.2 Angular Velocity

The angular velocity Ω with respect to the Schwarzschild coordinate time t is the rate measured with respect to a stationary clock at infinity, where t and the proper time of such a clock coincide.

Find $\Omega = \frac{d\phi}{dt}$ for a circular orbit of radius r by the following two steps:

1. Express $\Omega(r)$ with e, l, for any equatorial orbit.

2. Find $l\left(r\right)$ and $e\left(r\right)$ for a circular orbit and substitute into Ω.
3. Find the angular velocity of the particle with respect to the proper time $\frac{d\phi}{d\tau}$.
4. Find the 4-velocity of the circulating particle.

1.2.1 $\frac{d\phi}{dt}$ for equatorial orbit

We divide the coordinates velocities and use (2) and (3)

$$\Omega=\frac{d\phi}{dt}=\frac{\frac{d\phi}{d\tau}}{\frac{dt}{d\tau}}=\frac{1}{r^2}\left(1-\frac{2M}{r}\right)\frac{l}{e} \tag{8}$$

1.2.2 $l\left(r\right)$, $e\left(r\right)$ and $\frac{d\phi}{dt}$ for circular orbit

We find l for circular orbit by setting the effective potential (7) derivative to zero

$$\frac{dV_{eff}}{dr}=\frac{M}{r}-\frac{l^2}{r^3}+\frac{3Ml^2}{r^4}=0 \tag{9}$$

$$Mr^2-l^2r+3Ml^2=0 \tag{10}$$

$$l^2=\frac{Mr^2}{r-3M}=Mr\left(1-\frac{3M}{r}\right)^{-1} \tag{11}$$

l for circular orbit

We find e from the energy equation (5),(6), by setting $\frac{dr}{d\tau}=0$ and plug (11) in

to V_{eff} (7). First,

$$1+\frac{l^2}{r^2}=\frac{r-2M}{r-3M}=\left(1-\frac{2M}{r}\right)\left(1-\frac{3M}{r}\right)^{-1} \tag{12}$$

Therefore

$$\frac{e^2-1}{2}=\frac{1}{2}\left[\left(1-\frac{2M}{r}\right)\left(1+\frac{l^2}{r^2}\right)-1\right] \tag{13}$$

$$e^2=\left(1-\frac{2M}{r}\right)^2\left(1-\frac{3M}{r}\right)^{-1} \tag{14}$$

e for circular orbit

We need the ratio $\frac{l}{e}$

$$\left(\frac{l}{e}\right)^2=Mr\left(1-\frac{2M}{r}\right)^{-2} \tag{15}$$

Now plug (15) into (8) yields

$$\Omega^2 = \frac{M}{r^3} \tag{16}$$

t angular velocity for circular orbit - Kepler law

1.2.3 $\frac{d\phi}{d\tau}$ for circular orbit

From (3) we have

$$\frac{d\phi}{d\tau} = \frac{l}{r^2} \tag{17}$$

Plug (11) for circular orbit

$$\left(\frac{d\phi}{d\tau}\right)^2 = \frac{M}{r^3}\left(1 - \frac{3M}{r}\right)^{-1} \tag{18}$$

proper angular velocity for circular orbit

This is faster than (16). A clock on the circulating orbit runs slow compared to a clock at infinity both because it is moving (time dilation) and because it is in a lower gravitational potential. Note that there can be no circular orbits with $r < 3M$.

The problem can also be done by computing

$$\frac{d\phi}{d\tau} = \frac{d\phi}{dt}\frac{dt}{d\tau} \tag{19}$$

where from (2) and (14)

$$\frac{dt}{d\tau} = e\left(1 - \frac{2M}{r}\right)^{-1} = \left(1 - \frac{3M}{r}\right)^{-\frac{1}{2}} \tag{20}$$

1.2.4 Four velocity of a particle in circular orbit

From (18), (20) and (16) we have

$$u^\mu = \left(\frac{dt}{d\tau}, 0, 0, \frac{d\phi}{d\tau}\right) = \left(1 - \frac{3M}{r}\right)^{-\frac{1}{2}}(1, 0, 0, \Omega) \tag{21}$$

Four velocity of a particle in circular orbit

1.3 Satellite Synchronization

A satellite orbit a planet of radius R in a circular motion. What does the orbit radius r should be so that the clock on the satellite would be synchronized with a stationary clock on the surface of the planet?

The proper time along a circular orbit (in equatorial plane) is

$$d\tau_{circ}^2 = \left(1 - \frac{2M}{r}\right) dt^2 - r^2 d\phi^2 \tag{22}$$

$$\begin{aligned} d\tau_{circ}^2 &= \left(1 - \frac{2M}{r}\right) dt^2 - r^2 \left(\frac{d\phi}{dt}\right)^2 dt^2 \\ &= \left(1 - \frac{2M}{r} - r^2\Omega^2\right) dt^2 \end{aligned} \tag{23}$$

Use result (16), the proper time of the satellite is

$$d\tau_r^2 = \left(1 - \frac{3M}{r}\right) dt^2 \tag{24}$$

We could have use instead result (20).

The proper time of the stationary observer is

$$d\tau_R^2 = \left(1 - \frac{2M}{R}\right) dt^2 \tag{25}$$

Synchronization condition

$$d\tau_r^2 = d\tau_R^2 \tag{26}$$

yields

$$r = \frac{3}{2}R \tag{27}$$

Notice that this is exactly same result we found from the weak field metric.

1.4 Measured Linear Velocity

Find the linear velocity of a particle in a circular orbit of radius r that would be measured by a stationary observer stationed at one point on the orbit. What is its value at the ISCO?

To find the measured velocity v we use

$$-u_{ob} \cdot u_p = \gamma \tag{28}$$

where

$$\gamma = \left(1 - v^2\right)^{-\frac{1}{2}} \tag{29}$$

u_p we found in (21)

$$u_p^\mu = \left(1 - \frac{3M}{r}\right)^{-\frac{1}{2}} (1, 0, 0, \Omega) \tag{30}$$

and the stationary observer four velocity is (4)

$$u_{ob}^\mu = \left(\left(1 - \frac{2M}{r}\right)^{-\frac{1}{2}}, 0, 0, 0\right) \tag{31}$$

(28) becomes

$$\begin{aligned} \gamma = -g_{tt} u_{ob}^t u_p^t &= \left(1 - \frac{2M}{r}\right)\left(1 - \frac{2M}{r}\right)^{-\frac{1}{2}}\left(1 - \frac{3M}{r}\right)^{-\frac{1}{2}} \\ &= \left(1 - \frac{2M}{r}\right)^{\frac{1}{2}}\left(1 - \frac{3M}{r}\right)^{-\frac{1}{2}} \end{aligned} \tag{32}$$

$\Rightarrow$

$$1 - v^2 = \frac{1 - \frac{3M}{r}}{1 - \frac{2M}{r}} \tag{33}$$

$\Rightarrow$

$$v^2 = 1 - \frac{1 - \frac{3M}{r}}{1 - \frac{2M}{r}} = \frac{\frac{M}{r}}{1 - \frac{2M}{r}} \tag{34}$$

$$\boxed{v = \left(\frac{M}{r}\right)^{1/2}\left(1 - \frac{2M}{r}\right)^{-1/2}} \tag{35}$$

ISCO (*Innermost stable circular orbit*) is

$$R_{ISCO} = 6M \tag{36}$$

Plug in (35) we find

$$v_{ISCO} = \frac{1}{2} \tag{37}$$

1.5 Perturbation Of A Circular Orbit

A small perturbation of an unstable circular orbit will grow exponentially in time, and a small perturbation of a stable circular orbit will oscillate harmonically in time. Show that a small displacement Δr from the unstable/stable maximum/minimum of the potential r_m will change as $\Delta r \propto e^{\omega\tau}$ and $\Delta r \propto cos\,(\omega\tau)$ respectively.

Differentiate the energy equation (5) with respect to τ yields

$$0 = \frac{dr}{d\tau}\frac{d^2r}{d\tau^2} + \frac{dV_{eff}}{dr}\frac{dr}{d\tau} \tag{38}$$

which is the standard force-acceleration equation

$$\frac{d^2r}{d\tau^2} = -\frac{dV_{eff}}{dr} \tag{39}$$

We look at

$$r = r_m + \Delta r \tag{40}$$

and expand the potential around r_m to leading order in Δr

$$V_{eff}(r) = V_{eff}(r_m) + \frac{1}{2}\frac{d^2V_{eff}}{dr^2}(r_m)(\Delta r)^2 \tag{41}$$

where $\frac{dV_{eff}}{dr}(r_m) = 0$ for circular orbits. The force to leading order is

$$\frac{dV_{eff}}{dr} = \frac{d^2V_{eff}}{dr^2}(r_m)\,\Delta r \tag{42}$$

Plug (42) and (40) in (39)

$$\frac{d^2(\Delta r)}{d\tau^2} = -\frac{d^2V_{eff}}{dr^2}(r_m)\,\Delta r \tag{43}$$

where we used $\frac{dr_m}{d\tau} = 0$ for circular orbits.

For perturbation around the maximum, $\frac{d^2V_{eff}}{dr^2}(r_m) < 0$, so denote

$$\omega^2 = -\frac{d^2V_{eff}}{dr^2}(r_m) \tag{44}$$

and the equation of motion for the perturbation (43) is

$$\frac{d^2(\Delta r)}{d\tau^2} = \omega^2\Delta r \tag{45}$$

with a solution

$$\Delta r \propto e^{\omega\tau} \tag{46}$$

Perturbation around a maximum

For perturbation around the minimum, $\frac{d^2 V_{eff}}{dr^2}(r_m) > 0$, so denote

$$\omega^2 = \frac{d^2 V_{eff}}{dr^2}(r_m) \tag{47}$$

and the equation of motion for the perturbation (43) is

$$\frac{d^2(\Delta r)}{d\tau^2} = -\omega^2 \Delta r \tag{48}$$

with a solution

$$\Delta r \propto cos(\omega\tau) \tag{49}$$

Perturbation around a minimum

2 Light Rays In Schwarzschild Geometry

2.1 The Basic Equations

The metric and the conserved quantities are (1), (2), (3).

The *impact parameter* is

$$b = \frac{l}{e} \tag{50}$$

The energy equation for a massive particle is

$$\frac{1}{b^2} = \frac{1}{l^2}\left(\frac{dr}{d\lambda}\right)^2 + W_{eff}(r) \tag{51}$$

where the effective potential for massless particles is

$$W_{eff}(r) = \frac{1}{r^2}\left(1 - \frac{2M}{r}\right) = \frac{1}{r^2} - \frac{2M}{r^3} \tag{52}$$

The potential has one stationary point, a maximum at

$$r_{max} = 3M \tag{53}$$

with value

$$W_{eff}(r_{max}) = \frac{1}{27M^2} \tag{54}$$

2.2 How Much Light Escapes To Infinity?

A stationary observer stationed at a radius $2M < r < 3M$ sends out light rays in various directions in the equatorial plane, making angles ψ with the radial direction. Radial light rays with $\psi = 0$ escape. What is the critical angle ψ_c beyond which the light rays will fall into the center of attraction?

A light four vector u is sent with an angle ψ from the radial direction. Let us use an orthonormal basis for the stationary observer $\left(e_{\hat{t}}, e_{\hat{r}}, e_{\hat{\theta}}, e_{\hat{\phi}}\right)$, which points in the direction of the coordinate lines. The angle ψ is

$$tan\left(\psi\right) = \frac{u^{\hat{\phi}}}{u^{\hat{r}}} = \frac{u \cdot e_{\hat{\phi}}}{u \cdot e_{\hat{r}}} \tag{55}$$

We will calculate the dot products in (55) in the coordinate basis, where we have equations for conserved quantities and an energy equation. The components of the orthonormal basis vectors in coordinate basis are

$$e^{\mu}_{\hat{\phi}} = \left(0, 0, 0, \frac{1}{r}\right) \tag{56}$$

$$e^{\mu}_{\hat{r}} = \left(0, \left(1 - \frac{2M}{r}\right)^{\frac{1}{2}}, 0, 0\right) \tag{57}$$

Therefore

$$u^{\hat{\phi}} = u \cdot e_{\hat{\phi}} = g_{\phi\phi} u^{\phi} e^{\phi}_{\hat{\phi}} = \left(r^2\right)\left(\frac{l}{r^2}\right)\frac{1}{r} = \frac{l}{r} \tag{58}$$

where we used (1), (3) and (56). (58) is basically "angular momentum equals radius times tangential velocity".

$$u^{\hat{r}} = u \cdot e_{\hat{r}} = g_{rr} u^r e^r_{\hat{r}} = \left(1 - \frac{2M}{r}\right)^{-1} \frac{dr}{d\lambda} \left(1 - \frac{2M}{r}\right)^{\frac{1}{2}} = \left(1 - \frac{2M}{r}\right)^{-\frac{1}{2}} \frac{dr}{d\lambda} \tag{59}$$

We can substitute $\frac{dr}{d\lambda}$ from the energy equation (51), (52)

$$\frac{dr}{d\lambda} = l\left(\frac{1}{b^2} - W_{eff}\left(r\right)\right)^{\frac{1}{2}} \tag{60}$$

Plugin also (52),

$$u^{\hat{r}} = \left(1 - \frac{2M}{r}\right)^{-\frac{1}{2}} l\left(\frac{1}{b^2} - \frac{1}{r^2}\left(1 - \frac{2M}{r}\right)\right)^{\frac{1}{2}} \tag{61}$$

The critical angle ψ_c below which light rays escape to infinity occurs when $\frac{1}{b^2} = \frac{1}{27M^2}$, which is the hight of the maximum potential (54). Then,

$$u_c^{\hat{r}} = \left(1 - \frac{2M}{r}\right)^{-\frac{1}{2}} l \left(\frac{1}{27M^2} - \frac{1}{r^2}\left(1 - \frac{2M}{r}\right)\right)^{\frac{1}{2}} \tag{62}$$

The critical angle (55) is, by (58) and (62)

$$tan\left(\psi_c\right) = \frac{1}{r}\left(1 - \frac{2M}{r}\right)^{\frac{1}{2}} \left(\frac{1}{27M^2} - \frac{1}{r^2}\left(1 - \frac{2M}{r}\right)\right)^{-\frac{1}{2}} \tag{63}$$

At $r = 3M$ $\frac{1}{b^2} = W_{eff}$ and $tan\left(\psi_c\left(r = 3M\right)\right) = \infty$, $\psi_c\left(r = 3M\right) = \frac{\pi}{2}$. There is a circular orbit for light at this radius, and this light is on the edge between escaping to infinity and falling inside.

At $r = 2M$ $tan(\psi_c\left(r = 2M\right)) = 0$, $\psi_c\left(r = 2M\right) = 0$. Only radial rays escape from this radius. From $r < 2M$ nothing can escape...