כבידה 1

2021/22 סמסטר א'

פתרון תרגיל 5

1. חרוט אור

(a) In order to find out the light cone we take $ds^2 = 0$ and so we get

$$dv\,(-udv + 2du) = 0$$

which has two solutions:

$$\begin{aligned} dv &= 0,\ v = const \\ \frac{dv}{du} &= \frac{2}{x},\ v = 2ln|u| + const \end{aligned}$$

(b) We draw the these two solutions to find the light cones:

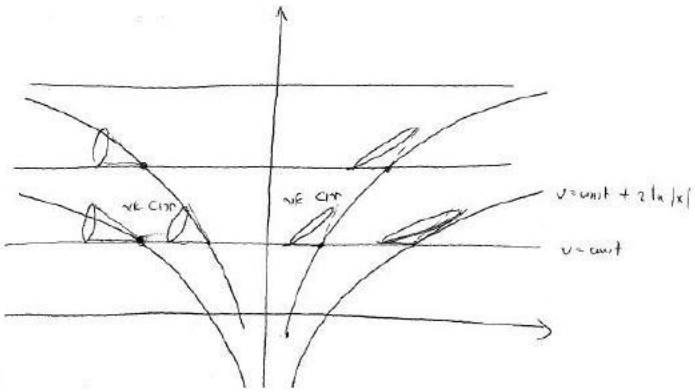


Figure 1:

(c) For a material particle $ds^2 < 0$ and so $dv(-udv + 2du) < 0$ and so $du < \frac{1}{2}udv$. For $u > 0$, $du < \frac{1}{2}udv > 0$ and so u can either increase or decrease. For $u < 0$, $du < \frac{1}{2}udv < 0$ and u can only decrease. That is, on the right of the v axis it is always timelike, on the left of the v axis it is . always spacelike, and so the horizon is at $u = 0$

2. משטח ונורמל

(a) A flat geometry with azimuthal symmetry is $ds^2 = dr^2 + dz^2 + r^2 d\phi^2$ and so we need to find a function $z = z(r)$ which will give this surface:

$$ds^2 = dr^2 \left(1 + \left(\frac{dz}{dr}\right)^2\right) + r^2 d\phi^2.$$

The equation is:

$$\begin{aligned} 1 + z'^2 &= \frac{1}{1 - 2M/r} \Rightarrow z'^2 = \frac{2M}{r - 2M} \\ \Rightarrow z &= \int \sqrt{\frac{2M}{r - 2M}} dr = s\sqrt{2M(r - 2M)} \end{aligned}$$

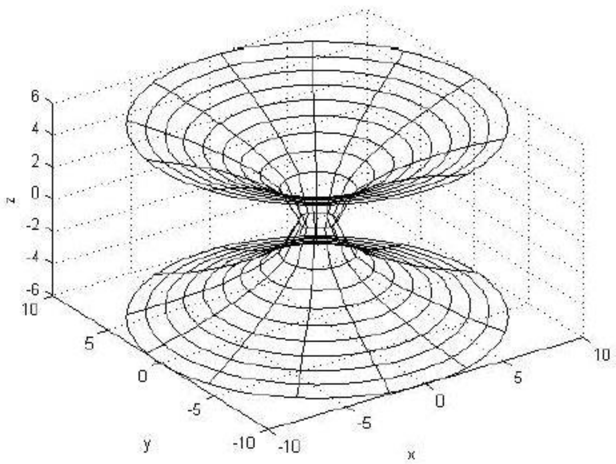


Figure 2:

(b) We can write the equation for the surface as $f(x^\alpha) = z - 2\sqrt{2M(r - 2M)} = 0$. Then the normal vector will be:

$$\begin{aligned} N_\mu &= f_{,\mu} \\ N_z &= 1 \\ N_r &= -\sqrt{\frac{2M}{r - 2M}} \\ N_\phi &= 0 \end{aligned}$$

The vector must be normalized:

$$\begin{aligned} g^{\alpha\beta} N_\alpha N_\beta &= N_z^2 + N_R^2 = 1 + \frac{2M}{r - 2M} = \frac{r}{r - 2M} \\ n_\mu &= \frac{N_\mu}{\sqrt{g^{\alpha\beta} N_\alpha N_\beta}}, \\ \vec{n} &= \left(\sqrt{1 - \frac{2M}{r}}, -\sqrt{\frac{2M}{r}}, 0\right) \end{aligned}$$

3. אור בכבידה ניוטונית

For light, $ds^2 = 0$. Write the metric and divide by $d\tau^2$:

$$\frac{ds^2}{d\tau^2} = -\left(1-\frac{r_s}{r}\right)\frac{dt^2}{d\tau^2} + \left(1+\frac{r_s}{r}\right)\left(\frac{dr^2}{d\tau^2} + r^2\frac{d\phi^2}{d\tau^2}\right).$$

Plug in the conserved charges:

$$\begin{aligned} E &= \left(1-\frac{r_s}{r}\right)\frac{dt}{d\tau},\ J = \left(1+\frac{r_s}{r}\right)\left(r^2\frac{d\phi}{d\tau}\right) \\ \frac{ds^2}{d\tau^2} &= \frac{-E^2}{1-\frac{r_s}{r}} + \left(1-\frac{r_s}{r}\right)\frac{dr^2}{d\tau^2} + \frac{J^2}{r^2\left(1+\frac{r_s}{r}\right)}. \end{aligned}$$

Multiply by -1 and use $ds^2 = 0$. Since it's a weak field:

$$0 \approx E^2\left(1+\frac{r_s}{r}\right) - \left(1+\frac{r_s}{r}\right)\frac{dr^2}{d\tau^2} - \left(1-\frac{r_s}{r}\right)\frac{J^2}{r^2}$$

and drop the term $\sim r^{-3}$ for J^2. Then to first order

$$E^2 - \left(\frac{dr}{d\tau}\right)^2 - \frac{J^2}{r^2} = 0$$

and this is the path in flat space. In a Newtonian metric you see that there is *no deflection* of light. The effect is only relativistic.

4. משפט נתר

We know $\frac{\partial L}{\partial x^1} = 0$.

$$S = \int d\sigma \sqrt{-g_{\alpha\beta}\frac{dx^\alpha}{d\sigma}\frac{dx^\beta}{d\sigma}}.$$

Euler Lagrange equations for $x^1$are

$$\frac{d}{d\sigma}\frac{\partial L}{\partial \frac{\partial x^1}{\partial\sigma}} = \frac{\partial L}{\partial x^1} = 0$$

(from the constraint).

$$\begin{aligned}\frac{\partial L}{\partial \frac{\partial x^\alpha}{\partial\sigma}} &= \frac{-g_{\alpha\beta}\frac{dx^\beta}{d\sigma}}{\sqrt{-g_{\alpha\beta}\frac{dx^\alpha}{d\sigma}\frac{dx^\beta}{d\sigma}}.} = \frac{-g_{\alpha\beta}\frac{dx^\beta}{d\sigma}}{L} \\ &= -g_{\alpha\beta}\frac{dx^\beta}{d\sigma}\frac{d\sigma}{d\tau} = -g_{\alpha\beta}\frac{dx^\beta}{d\tau}.\end{aligned}$$

For x^1 this becomes

$$\begin{aligned}\frac{\partial L}{\partial \frac{\partial x^1}{\partial\sigma}} &= -g_{1\beta}\frac{dx^\beta}{d\tau} \\ \frac{d}{d\tau}\frac{\partial L}{\partial \frac{\partial x^1}{\partial\sigma}} &= -g_{1\beta}\frac{d^2x^\beta}{d\tau^2}.\end{aligned}$$

The Killing vector

$$\xi = (0,1,0,0)$$

and so

$$-g_{1\beta} = -g_{\alpha\beta}\xi^\alpha$$

(the projection of the metric on ξ), and so

$$\frac{\partial L}{\partial \frac{\partial x^1}{\partial\sigma}} = -g_{\alpha\beta}\xi^\alpha\frac{dx^\beta}{d\tau} = -\xi \cdot u = const$$

and so a conserved quantity $\partial x^1/\partial\sigma$ has given us a constant.

5. רדיוס שוורצשילד

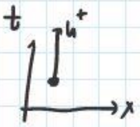
$v^\mu = (0, R_s, 0, 0)$ and the magnitude of any vector is $\sqrt{g_{\alpha\alpha}x^\alpha x^\alpha}$. We want to know when

$$g_{11}R_s^2 = \left(\frac{1}{1-\frac{2M}{r}}\right)R_s^2 = 0.$$

Never! For $r = 2M$ (at the black hole horizon) it will be infinite .

6. כוח של חללית מחוץ לכוכב

$$f^\alpha = \frac{\partial u^\alpha}{\partial \tau} + \Gamma^\alpha_{\beta\gamma} u^\beta u^\gamma$$



$$u^i = 0 \qquad dr = d\theta = d\varphi = 0$$

$$\frac{\partial u^\alpha}{\partial \tau} = \frac{\partial u^\alpha}{\partial t}\frac{\partial t}{\partial \tau} = 0$$

[illegible] בתנועה:

$$\left\{ \frac{\partial u^t}{\partial \tau} = \frac{\partial u^t}{\partial r}\underbrace{\frac{\partial r}{\partial \tau}}_{0} = 0 \right.$$

$$u^t = ? \qquad u \cdot u = -1 \Rightarrow g_{tt}(u^t)^2 = -1 \Rightarrow u^t = \frac{1}{\sqrt{1-\frac{2m}{r}}}$$

$$\boxed{f^\alpha = \Gamma^\alpha_{00}(u^0)^2} \qquad \alpha =$$

$$\alpha = 0 \Rightarrow \Gamma^0_{00} = \frac{1}{2} g^{00}(\partial_0 g \dots - \partial_0 g) = 0$$

$$\Rightarrow f^t = 0$$

$$\alpha = r \Rightarrow \Gamma^r_{00} = \frac{1}{2} g^{rr}(\partial_0 g_{r0} + \partial_0 g_{0r} - \partial_r g_{00}) \qquad \left\{ +\partial_r\left(1-\frac{2m}{r}\right) = -\frac{2m}{r^2} \right.$$

$$= \frac{1}{2}\left(1-\frac{2m}{r}\right)\frac{2m}{r^2} = \frac{m}{r^2}\left(1-\frac{2m}{r}\right)$$

$$f^r = \Gamma^r_{00}(u^0)^2 = \frac{m}{r^2}\left(1-\frac{2m}{r}\right)\underbrace{\left(1-\frac{2m}{r}\right)^{-1}}_{(u^0)^2} = \frac{m}{r^2}$$

$$f^\alpha = \left(0, \frac{m}{r^2}, 0, 0\right)$$

$$F = \sqrt{f \cdot f} = \sqrt{g_{\alpha\beta} f^\alpha f^\beta} = \sqrt{\left(1-\frac{2m}{r}\right)^{-1}\left(\frac{m}{r^2}\right)^2} = \frac{m}{r^2}\frac{1}{\sqrt{1-\frac{2m}{r}}}$$