Exponentially growing/decaying steps

Consider

$$a_m = a^m \qquad \text{with } a = const.$$

For $a > 1$ the steps are growing while for $a < 1$ the steps are decaying.

$$C_{N,l} = c_l \sum_{m=1}^{N} a_m^l = c_l \sum_{m=1}^{N} a^{ml} = c_l \left(\frac{1 - a^{l(N+1)}}{1 - a^l} - 1 \right) = c_l a^l \left(\frac{1 - a^{lN}}{1 - a^l} \right).$$

The asymptotic behavior of the cumulants is given by

$$C_{N,l} \sim \begin{cases} c_l \frac{a^l}{1-a^l} & a < 1 \\ c_l \frac{a^{l(N+1)}}{a^l - 1} & a > 1 \end{cases}.$$

The common step $\widetilde{\Delta x}$ may have different PDFs. In what follows we will consider two examples.

Gaussian steps with $std = \sigma$

We already showed that the convolution of Gaussian PDFs yields a Gaussian PDF. Therefore, we know the exact PDF of the position after N steps.

$$\ln \left(\widetilde{p}_N (k) \right) = -\frac{k^2}{2} C_{N,2} = -\frac{k^2}{2} \sigma^2 a^2 \left(\frac{1 - a^{2N}}{1 - a^2} \right);$$

Defining a scaled variable

$$\xi = \frac{x_N}{\sigma a \sqrt{\frac{1-a^{2N}}{1-a^2}}};$$

we find that its PDF is

$$\begin{aligned} p_\xi (\xi) &= \frac{1}{\sqrt{2\pi}} e^{-\frac{\xi^2}{2}}; \\ p_x (x_N) &= \frac{1}{\sigma a \sqrt{\frac{1-a^{2N}}{1-a^2}} \sqrt{2\pi}} e^{-\frac{x^2}{2\sigma^2 a^2 \frac{1-a^{2N}}{1-a^2}}}; \\ \langle x_N^2 \rangle &= \sigma^2 a^2 \frac{1 - a^{2N}}{1 - a^2} \sim \begin{cases} \sigma^2 \frac{a^{2(N+1)}}{a^2 - 1} & a > 1 \\ \frac{\sigma^2 a^2}{1 - a^2} & a < 1 \end{cases}. \end{aligned}$$

For $a > 1$ we have anomalous diffusion and the MSD grows exponentially with the number of steps while for $a < 1$ the MSD is bounded and does not grow indefinitely with the number of steps. However, for all values of a the PDF is Gaussian.

A Simple Walk with Decaying Steps

In this example we consider a simple RW with decaying steps. Namely,

$$x_N = \sum_{m=1}^{N} a^m \varepsilon_m;$$

with $a < 1$ and ε_m being a random variable taking the values 1 and -1 with the same probability (1/2). The PDF of the $m'th$ step is given by

$$p_{x_m} (x_m) = \frac{1}{2} \left(\delta \left(x_m - a^m \right) + \delta \left(x_m + a^m \right) \right).$$

The corresponding characteristic function is:

$$\widetilde{p}_{x_m}(k) = \cos(ka^m).$$

From the convolution theorem we find

$$\widetilde{p}_N(k,a) = \prod_{m=1}^{N} \cos(ka^m) = \cos(ka)\cos\left(ka^2\right)...\cos\left(ka^N\right).$$

We can also see that

$$\widetilde{p}_N(k,a) = \widetilde{p}_{ceiling(N/2)}\left(k/a,a^2\right)\widetilde{p}_{floor(N/2)}\left(k,a^2\right) = \prod_{m=1}^{ceiling(N/2)} \cos\left(\frac{k}{a}a^{2m}\right)\prod_{r=1}^{floor(N/2)} \cos\left(ka^{2r}\right),$$

and, in particular,

$$\begin{aligned}
\widetilde{p}_\infty(k,a) &= \widetilde{p}_\infty\left(k/a,a^2\right)\widetilde{p}_\infty\left(k,a^2\right) = \prod_{m=1}^{\infty} \cos\left(\frac{k}{a}a^{2m}\right)\prod_{r=1}^{\infty} \cos\left(ka^{2r}\right). \\
\widetilde{p}_\infty(k,a) &= \widetilde{p}_\infty\left(k/a,a^2\right)\widetilde{p}_\infty\left(k,a^2\right) = \widetilde{p}_\infty\left(k/a^2,a^3\right)\widetilde{p}_\infty\left(k/a,a^3\right)\widetilde{p}_\infty\left(k,a^2\right) \\
&= \widetilde{p}_\infty\left(k/a^2,a^3\right)\widetilde{p}_\infty\left(k/a,a^3\right)^2\widetilde{p}_\infty\left(k,a^3\right)
\end{aligned}$$

For the case of $a = 1/2$ we can solve exactly.

$$\begin{aligned}
\widetilde{p}_N(k,1/2) &= \prod_{m=1}^{N} \cos\left(k\frac{1}{2^m}\right) = \cos\left(k\frac{1}{2^1}\right)\cos\left(k\frac{1}{2^2}\right)\cos\left(k\frac{1}{2^3}\right)...\cos\left(k\frac{1}{2^N}\right) \\
&= \frac{\sin(k)}{2\sin(k/2)}\frac{\sin(k/2)}{2\sin(k/4)}\frac{\sin(k/4)}{2\sin(k/8)}...\frac{\sin(k/2^{N-1})}{2\sin(k/2^N)} = \frac{\sin(k)}{2^N\sin(k/2^N)} \\
&\sim \frac{\sin(k)}{k}
\end{aligned}$$

We used

$$\cos(a/2) = \frac{\sin(a)}{2\sin(a/2)}.$$

The inverse Fourier transform of the sinc function is the uniform distribution in the interval $[-1,1]$,

$$p_\infty(x,1/2) = \begin{cases} 1/2 & -1 < x < 1 \\ 0 & else \end{cases}.$$

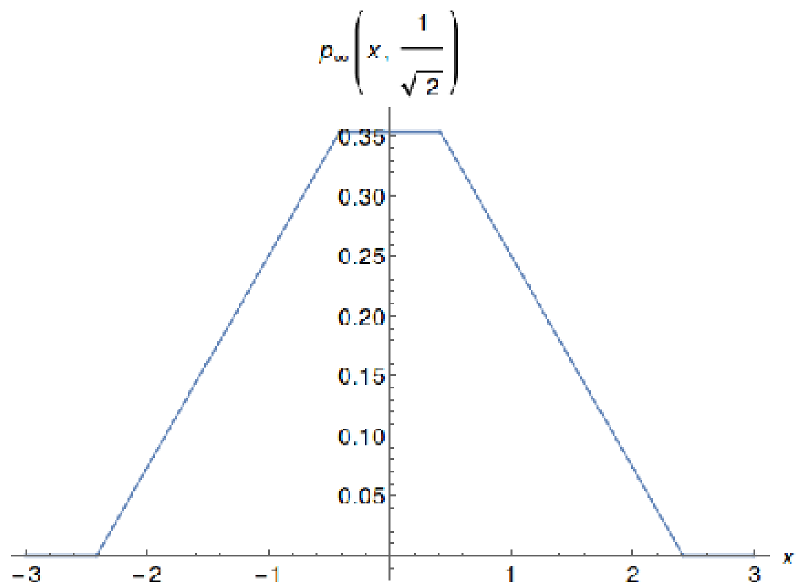
$$\frac{1}{2}\int_{-\infty}^{\infty} e^{-ikx}\Theta(1+x)\Theta(1-x)\,dx = \frac{1}{2}\int_{-1}^{1} e^{-ikx}dx = \frac{e^{-ik}-e^{ik}}{2(-ik)} = \frac{\sin(k)}{k}.$$

Since we know the PDF for $a = 1/2$ we can find the PDF for $a = 1/2^{1/m}$ (with m being an integer). For example for $a = 1/\sqrt{2}$ we can write

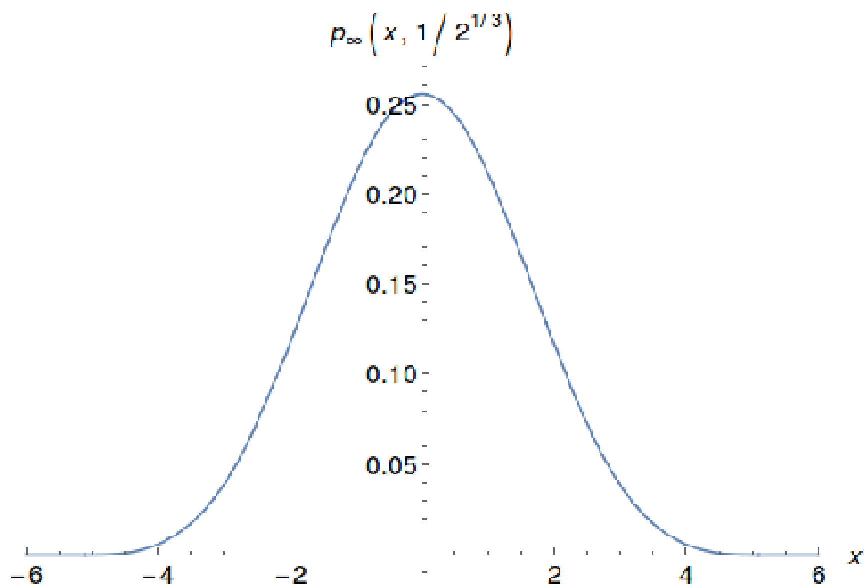
$$\begin{aligned}
\widetilde{p}_\infty\left(k,1/\sqrt{2}\right) &= \widetilde{p}_\infty\left(\sqrt{2}k,1/2\right)\widetilde{p}_\infty(k,1/2) \\
&\rightarrow\ p_\infty\left(x,1/\sqrt{2}\right) = p_\infty\left(\frac{x}{\sqrt{2}},1/2\right) * p_\infty(x,1/2).
\end{aligned}$$

(the $*$ denotes the convolution). Therefore, the PDF for $a = 1/\sqrt{2}$ is given by the convolution of two step functions, one in the interval $[-1,1]$ and one in the interval $[-\sqrt{2},\sqrt{2}]$.

$$p_\infty\left(x,1/\sqrt{2}\right) = \frac{1}{4\sqrt{2}}\int_{-\infty}^{\infty} \Theta(u-1)\Theta(u+1)\Theta\left(x-u-\sqrt{2}\right)\Theta\left(x-u+\sqrt{2}\right)du$$



p∞(x, 1/√2)
0.35
0.30
0.25
0.20
0.15
0.10
0.05
−3
−2
−1
1
2
3
x



p∞(x, 1/2^(1/3))
0.25
0.20
0.15
0.10
0.05
−6
−4
−2
2
4
6
x

Note that it is also possible to find a recursion relation without using the Fourier transform. Let's define as $p^s_\infty(x,a)$ the PDF for the case when the walker start their steps as a^2 rather than a (thus step sizes go like a^2, a^3...). This PDF should look like $p_\infty(x,a)$ except that the distance is scaled by a factor of a. The following relation between the probabilities hold

$$p_\infty\left(\frac{x}{a},a\right)\frac{dx}{a}=p^s_\infty(x,a)\,dx.$$

Now we should remember that $p_\infty(x,a)$ is just the average of $p^s_\infty(x-a,a)$ and $p^s_\infty(x+a,a)$ because after the first step the walker is either in $+a$ or in $-a$. Therefore we find

$$p_\infty(x,a)=\frac{1}{2}\left(p^s_\infty(x-a,a)+p^s_\infty(x+a,a)\right)=\frac{1}{2a}\left(p_\infty\left(\frac{x-a}{a},a\right)+p_\infty\left(\frac{x+a}{a},a\right)\right).$$

Another way to understand the above relation is to look at the Fourier transform

$$\widetilde{p}_\infty(k,a)=\prod_{m=1}^{\infty}\cos\left(ka^m\right)=\cos(ka)\prod_{m=1}^{\infty}\cos\left((ka)\,a^m\right)=\cos(ka)\,\widetilde{p}_\infty(ka,a)=\frac{1}{2}\left(e^{-ika}+e^{ika}\right)\widetilde{p}_\infty(ka,a),$$

which implies that

$$p_\infty(x,a)=\frac{1}{2a}\left(p_\infty\left(\frac{x-a}{a},a\right)+p_\infty\left(\frac{x+a}{a},a\right)\right).$$

Note that we could also write the more general relation for any N

$$\widetilde{p}_{N+1}(k,a)=\prod_{m=1}^{N+1}\cos\left(ka^m\right)=\cos(ka)\prod_{m=1}^{N}\cos\left((ka)\,a^m\right)=\cos(ka)\,\widetilde{p}_N(ka,a)=\frac{1}{2}\left(e^{-ika}+e^{ika}\right)\widetilde{p}_N(ka,a),$$

and

$$p_{N+1}(x,a)=\frac{1}{2a}\left(p_N\left(\frac{x-a}{a},a\right)+p_N\left(\frac{x+a}{a},a\right)\right).$$

Correlated walks

The correlation of two variables is defined as

$$\begin{aligned}\rho(x,y) &= \rho(y,x)=\frac{\langle(x-\langle x\rangle)(y-\langle y\rangle)\rangle}{\sigma_x\sigma_y};\\ \sigma_x^2 &= \left\langle(x-\langle x\rangle)^2\right\rangle.\end{aligned}$$

The correlation coefficient is in the range

$$-1\le\rho(x,y)\le 1.$$

****Proof

$$\begin{aligned}Var(tx+y) &= \left\langle(tx+y-\langle tx+y\rangle)^2\right\rangle\ge 0\\ Var(tx+y) &= \left\langle(tx+y)^2-\langle tx+y\rangle^2\right\rangle=t^2\left\langle x^2\right\rangle+\left\langle y^2\right\rangle+2t\langle xy\rangle-t^2\langle x\rangle^2-\langle y\rangle^2-2t\langle x\rangle\langle y\rangle\\ &= t^2Var(x)+Var(y)+2tCov(x,y)\end{aligned}$$

Denoting

$$\begin{aligned}A &= Var(x);\\ B &= 2Cov(x,y);\\ C &= Var(y);\end{aligned}$$

we find that the quadratic polynomial of t is always non-negative and that occurs just if the discriminant is non-positive, i.e.,

$$\begin{aligned} B^2 - 4AC &\leq 0; \\ 4Cov\,(x,y)^2 - 4Var(x)Var(y) &\leq 0; \\ Cov\,(x,y)^2 &\leq Var(x)Var(y) \\ &\rightarrow -1 \leq \rho\,(x,y) \leq 1. \end{aligned}$$

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When $\rho = 0$ the variables are not correlated because $\langle \Delta x \Delta y \rangle = \langle \Delta x \rangle \langle \Delta y \rangle$. When $\rho = 1$, the variables are perfectly correlated, i.e., $\Delta x/\sigma_x$ is an exact copy of $\Delta y/\sigma_y$. When $\rho = -1$, the variables are perfectly anti-correlated, i.e., $\Delta x/\sigma_x$ is an exact copy of $-\Delta y/\sigma_y$.

Correlation function is defined as

$$C\,(m,m') = \langle \Delta \overrightarrow{x}_m \Delta \overrightarrow{x}_{m'} \rangle = Cov\,(\Delta \overrightarrow{x}_m, \Delta \overrightarrow{x}_{m'})\,,$$

where m, m' are the numbers of steps taken along the path of a random walker ($m \rightarrow m+1 \rightarrow m+2...m'$).

Denoting the total displacement as

$$\overrightarrow{X}_N = \sum_{m=1}^{N} \Delta \overrightarrow{x}_m,$$

we can write the average squared displacement as

$$\left\langle \overrightarrow{X}_N^2 \right\rangle = \left\langle \sum_{m=1}^{N} \Delta \overrightarrow{x}_m \sum_{m'=1}^{N} \Delta \overrightarrow{x}_{m'} \right\rangle = \sum_{m=1}^{N} \sum_{m'=1}^{N} \langle \Delta \overrightarrow{x}_m \Delta \overrightarrow{x}_{m'} \rangle = \sum_{m=1}^{N} \sum_{m'=1}^{N} C\,(m,m')\,.$$

If we assume that the correlation function is translation invariant, i.e.,

$$C\,(m,m') = C\,(|m-m'|) \qquad \text{or} \qquad C\,(m) = \langle \Delta \overrightarrow{x}_{m+m_0} \Delta \overrightarrow{x}_{m_0} \rangle \quad \text{for any} \quad m_0,$$

we can write the MSD as

$$\begin{aligned} \left\langle \overrightarrow{X}_N^2 \right\rangle &= \sum_{m=1}^{N} \sum_{m'=1}^{N} C\,(m,m') = NC\,(0) + 2\,(N-1)\,C\,(1) + 2\,(N-2)\,C\,(2) + ...2\,(1)\,C\,(N-1) \\ &= NC\,(0) + 2\sum_{k=1}^{N-1} (N-k)\,C\,(k)\,. \end{aligned}$$

In the continuum limit when $t' = k\tau$ and $t = N\tau$,

$$\frac{C\,(m)}{\tau^2} = \left\langle \frac{\Delta \overrightarrow{x}_{m_0}}{\tau} \cdot \frac{\Delta \overrightarrow{x}_{m+m_0}}{\tau} \right\rangle = \langle \overrightarrow{v}\,(0) \cdot \overrightarrow{v}\,(t) \rangle\,,$$

the velocity auto-correlation function. Using this function we can express the MSD as

$$\left\langle \overrightarrow{X}_t^2 \right\rangle = \left\langle \overrightarrow{v}^2\,(0) \right\rangle t\tau + 2\int_0^t \langle \overrightarrow{v}\,(0) \cdot \overrightarrow{v}\,(t') \rangle\,(t-t')\,dt'$$

We define the diffusion coefficient as

$$D = \lim_{t \to \infty} \frac{1}{2}\frac{d}{dt} \left\langle \overrightarrow{X}_t^2 \right\rangle = \left\langle \overrightarrow{v}^2\,(0) \right\rangle \tau + \int_0^t \langle \overrightarrow{v}\,(0) \cdot \overrightarrow{v}\,(t') \rangle\,dt'.$$

If the correlation time of the velocity is much larger than the time between consecutive steps we can neglect the first term and get the Kubo formula

$$D = \int_0^t \langle \overrightarrow{v}(0) \cdot \overrightarrow{v}(t') \rangle \, dt'.$$

Back to the RW, the asymptotic dependence of D on the number of steps determines the type of diffusion we get.

If $D \sim const$ the RW corresponds to normal diffusion. $\left\langle \overrightarrow{X}_N^2 \right\rangle \sim N$ or $\left\langle \overrightarrow{X}_t^2 \right\rangle \sim (2d)Dt$.

If D ~diverges, the RW corresponds to super-diffusion, i.e., $\left\langle \overrightarrow{X}_N^2 \right\rangle \sim N^{1+\mu}$ with $0 < \mu \leq 1$ or $\left\langle \overrightarrow{X}_t^2 \right\rangle \sim t^{1+\mu}$.

If D ~vanishes, the RW corresponds to sub-diffusion, i.e., $\left\langle \overrightarrow{X}_N^2 \right\rangle \sim N^{\mu}$ with $0 < \mu \leq 1$ or $\left\langle \overrightarrow{X}_t^2 \right\rangle \sim t^{\mu}$.