Example: Exponentially decaying correlation

Suppose

$$C(n) = \rho^n \quad \text{or} \quad C(n) = e^{-2n/n_c} \quad \text{with } n_c = -2/\log\rho \text{ and } 0 \le \rho \le 1.$$

If $\rho \to 1$, $n_c >> 1$ and we can think of the walk as a walk of n/n_c independent steps with each of these steps scaled as $n_c\sigma$ (where σ is the STD of a single step in the original walk). We might consider a renormalized MSD $\widetilde{\langle X^2 \rangle} = \langle X_n^2 \rangle / \left(n_c^2 \sigma^2\right)$ and renormalized number of steps $\widetilde{n} = n/n_c$. From dimensional arguments we can find that

$$\widetilde{\langle X^2 \rangle} = \phi\left(\widetilde{n}, \frac{1}{n_c}\right).$$

To get ϕ we calculate $\langle X_n^2 \rangle$ (for simplicity we assume $\sigma = 1$)

$$\begin{aligned}
\langle X_n^2 \rangle &= nC(0) + 2\sum_{k=1}^{n-1}(n-k)C(k) = n + 2\sum_{k=1}^{n-1}(n-k)\rho^k = -n + 2\sum_{k=0}^{n}(n-k)\rho^k \\
&= -n + 2n\sum_{k=0}^{n}\rho^k - 2\sum_{k=0}^{n}k\rho^k = -n + 2n\sum_{k=0}^{n}\rho^k - 2\rho\frac{d}{d\rho}\sum_{k=0}^{n}\rho^k \\
&= -n + 2n\frac{1-\rho^{n+1}}{1-\rho} - 2\rho\frac{d}{d\rho}\frac{1-\rho^{n+1}}{1-\rho} \\
&= -n + 2n\frac{1-\rho^{n+1}}{1-\rho} - 2\frac{-(n+1)\rho^{n+1}(1-\rho) + \rho - \rho^{n+2}}{(1-\rho)^2} \\
&= n\frac{1+\rho}{1-\rho} + 2\rho\frac{(\rho^n - 1)}{(1-\rho)^2}
\end{aligned}$$

The asymptotic behavior of the MSD goes like

$$\langle X_n^2 \rangle \sim n,$$

which implies a normal diffusion with

$$D = D_0\frac{1+\rho}{1-\rho},$$

where D_0 is the diffusion coefficient in the absence of correlation, $\rho = 0$. If we consider the case of $\rho \to 1$, we may set $\rho = 1 - \varepsilon$ with $\varepsilon << 1$.

$$\begin{aligned}
n_c &= -2/\ln\rho = -2/\ln(1-\varepsilon) = 2/\varepsilon; \\
\frac{1+\rho}{1-\rho} &= \frac{2-\varepsilon}{\varepsilon} \sim \frac{2}{\varepsilon} = n_c. \\
2\rho\frac{(\rho^n - 1)}{(1-\rho)^2} &\sim 2\frac{e^{-2n/n_c} - 1}{\varepsilon^2} = \frac{n_c^2}{2}\left(e^{-2n/n_c} - 1\right).
\end{aligned}$$

$$\begin{aligned}
\widetilde{\langle X^2 \rangle} &= \frac{\langle X^2 \rangle}{n_c^2\sigma^2} \sim_{\rho\to 1} \frac{n}{n_c} + \frac{1}{2}\left(e^{-2n/n_c} - 1\right) = \widetilde{n} + \frac{1}{2}\left(e^{-2\widetilde{n}} - 1\right) = \phi(\widetilde{n}) \\
\phi(\widetilde{n}) &\sim \begin{cases} \widetilde{n} & \widetilde{n} >> 1 (n >> n_c) \quad \text{diffusive regime} \\ \widetilde{n}^2 & \widetilde{n} << 1 (n << n_c) \quad \text{ballistic regime} \end{cases}.
\end{aligned}$$

Renewal Processes

A renewal process is one in which the "events" take place at random times $t_i s$. Once an event happened the process "forgets" all its history and is being renewed. A measurement of such a process consists of a time series recording the event times

$$0 < t_1 < t_2 < t_3 .... < t_{n-1} < t_n < t < t_{n+1},$$

where n is the number of events occurred during the measurement time t. The waiting times are given by $\tau_i = t_i - t_{i-1}$, the backward recurrence time $B = t - t_n$ and the forward recurrence time $E = t_{n+1} - t$.

The number of events for a non-equilibrium renewal process

$W(n,t)$ is the probability of n renewal events in the time interval $[0,t]$. We may express it as

$$W(n,t) = \langle I(t_n < t < t_{n+1}) \rangle, \tag{1}$$

where the indicator function is defined as

$$I(t_n < t < t_{n+1}) = \begin{cases} 1 & t_n < t < t_{n+1} \\ 0 & otherwise \end{cases}. \tag{2}$$

The indicator function ensures that we only "count" trajectories with n events and the average is over all possible realizations of random waiting times. The Laplace transform of $W(n,t)$ is

$$\widehat{W}(n,s) = \left\langle \int_0^\infty e^{-st} I(t_n < t < t_{n+1}) \, dt \right\rangle = \left\langle \frac{e^{-st_n} - e^{-st_{n+1}}}{s} \right\rangle. \tag{3}$$

Remembering that $t_n = \sum_{i=1}^{n} \tau_i$ and the fact that the waiting times are i.i.d., we write

$$\left\langle e^{-st_n} \right\rangle = \widehat{\psi}^n(s), \tag{4}$$

where

$$\widehat{\psi}(s) = \left\langle e^{-s\tau} \right\rangle = \int_0^\infty e^{-s\tau} \psi(\tau) \, d\tau. \tag{5}$$

Hence,

$$\widehat{W}(n,s) = \widehat{\psi}^n(s) \frac{1 - \widehat{\psi}(s)}{s}. \tag{6}$$

It is possible to derive the same result using a different approach. Let $Q_n(t)\,dt$ be the probability that the $n'th$ event takes place at the interval $[t, t+dt]$. Then we have

$$Q_{n+1}(t) = \int_0^t Q_n(t-\tau) \psi(\tau) \, d\tau. \tag{7}$$

The above equation means that the probability density of having the $(n+1)'th$ event at time t is equal to the sum of the probabilities to have the $n'th$ event happening at some earlier time $t-\tau$ and the $(n+1)'th$ event happening time τ later. In Laplace space the above equation may be written as

$$\widehat{Q}_{n+1}(s) = \widehat{Q}_n(s) \widehat{\psi}(s). \tag{8}$$

Assuming that the process starts at $t = 0$, namely, we start recording the time series exactly when event occurred, then we have $\widehat{Q}_1(s) = \widehat{\psi}(s)$, and therefore, $\widehat{Q}_n(s) = \widehat{\psi}^n(s)$. The probability that an event has not occurred up to time t is given by

$$R(t) = 1 - \int_0^t \psi(\tau)\, d\tau. \tag{9}$$

Therefore, the probability that n events occurred in the interval $[0, t]$ is

$$W(n,t) = \int_0^t Q_n(t-\tau)\, R(\tau)\, d\tau. \tag{10}$$

The equation above means that the $n'th$ event occurred at time $t - \tau$, earlier than t, and no event happened in the interval $[t-\tau, t]$. In Laplace space the equation above takes the form

$$\widehat{W}(n,s) = \widehat{Q}_n(s)\, \widehat{R}(s) = \widehat{\psi}^n(s)\, \frac{1-\widehat{\psi}(s)}{s}. \tag{11}$$

(where we used the Laplace transform of $R(t)$, $\widehat{R}(s) = \frac{1-\widehat{\psi}(s)}{s}$). Obviously, this result coincides with the result of the alternative derivation. An important assumption in the derivation above is that an event occurred at $t = 0$. This is known as a non-equilibrium renewal process.

It is easy to calculate $\langle \widehat{n}(s) \rangle$,

$$\begin{aligned}\langle \widehat{n}(s) \rangle &= \sum_{n=0}^{\infty} n\widehat{W}(n,s) = \frac{1-\widehat{\psi}(s)}{s}\sum_{n=0}^{\infty} n\widehat{\psi}^n(s) = \frac{1-\widehat{\psi}(s)}{s}\widehat{\psi}(s)\frac{\partial}{\partial\widehat{\psi}(s)}\sum_{n=0}^{\infty}\widehat{\psi}^n(s) \\ &= \frac{1-\widehat{\psi}(s)}{s}\frac{\widehat{\psi}(s)}{\left[1-\widehat{\psi}(s)\right]^2} = \frac{\widehat{\psi}(s)}{s\left[1-\widehat{\psi}(s)\right]}.\end{aligned} \tag{12}$$

In what follows we would like to distinguish between two cases,

$$case\ 1 \ : \ \widehat{\psi}(s) \sim 1 - As^{\alpha} \qquad 0 < \alpha < 1 \tag{13}$$

$$case\ 2 \ : \ \widehat{\psi}(s) \sim 1 - \langle \tau \rangle s \tag{14}$$

As we saw in the previous lectures, for *case* 1 the mean waiting time diverges while for *case* 2 it is finite. Substituting the asymptotic behavior in the expression for the average number of events we find

$$\langle \widehat{n}(s) \rangle \sim \begin{cases} \frac{1}{As^{\alpha+1}} & case\ 1 \\ \frac{1}{\langle \tau \rangle s^2} & case\ 2 \end{cases}. \tag{15}$$

Using our results for the asymptotic behavior of the Laplace forms above we find the long time limit behavior of the average number of events,

$$\langle n(t) \rangle \sim \begin{cases} \frac{t^{\alpha}}{A\Gamma(1+\alpha)} & case\ 1 \\ \frac{t}{\langle \tau \rangle} & case\ 2 \end{cases}. \tag{16}$$

The result of *case* 2 is easily understood (the simple law of large numbers). The result of *case* 1 can be understood by realizing that we have to replace $\langle \tau \rangle$ by $\int_0^t \tau\psi(\tau)\, d\tau \simeq t^{1-\alpha}$, hence, $\langle n(t) \rangle \sim t^{\alpha}$.

Now we turn to study the asymptotic behavior of $W(n,t)$ in the long time limit. We rewrite equation (6) as

$$\widehat{W}(n,s) = \widehat{\psi}^n(s)\, \frac{1-\widehat{\psi}(s)}{s} = \frac{1-\widehat{\psi}(s)}{s} e^{n\ln\left(\widehat{\psi}(s)\right)}.$$

Expanding in s we write it as:

$$\widehat{W}(n,s) \sim As^{\alpha-1}e^{-Ans^{\alpha}} = -\frac{1}{n\alpha}\frac{d}{ds}e^{-Ans^{\alpha}}.$$

Inverting the Laplace transform we find

$$W(n,t) \sim \frac{t}{n\alpha} l_{\alpha,An,1}(t) = \frac{t}{\alpha n^{1+1/\alpha} A^{1/\alpha}} l_{\alpha,1,1}\left(\frac{t}{[An]^{1/\alpha}}\right). \tag{17}$$

This function is known as the inverse Levy function. For the case $\alpha = 1/2$

$$W(n,t) \sim \frac{Ae^{-\frac{A^2n^2}{4t}}}{\sqrt{\pi t}}. \tag{18}$$

Note that the peak is at $n = 0$ while $\langle n(t)\rangle \propto t^{\alpha}$. The most likely n is far from the average. For *case* 2 in which the first moment is finite and $\widehat{\psi}(s) \sim 1 - \langle\tau\rangle s$ we find

$$\widehat{W}(n,s) \sim \langle\tau\rangle e^{-\langle\tau\rangle ns} \rightarrow W(n,t) \sim \langle\tau\rangle \delta(t - \langle\tau\rangle n). \tag{19}$$

Obviously the result above neglects the fluctuations. More generally we would expect to have a Gaussian distribution around the mean.

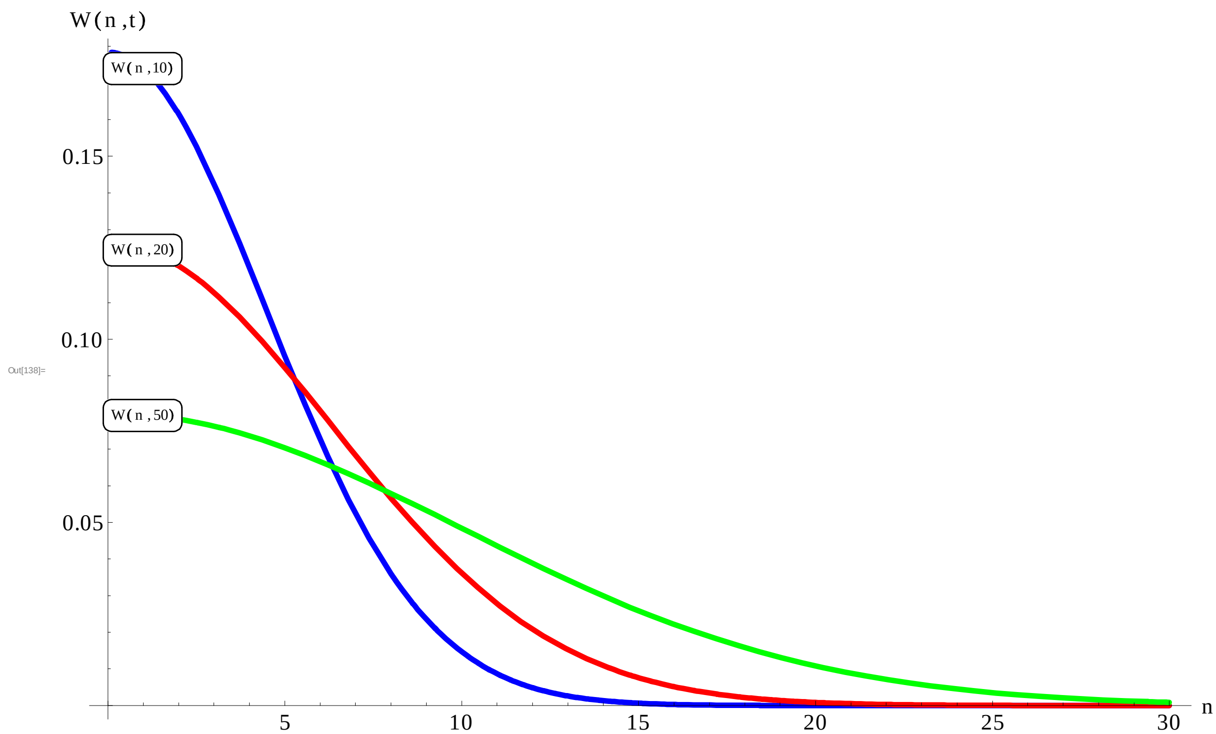

Figure 1: Figure 1: Plot of $W(n,t)$ versus n, with $\alpha = 1/2$. The different curves corrrespond to different times