Gravity 1 - Recitation 9

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Contents

1 Transformation of Tensor Components Between Coordinate Bases

1.1 Vector Transformation

Given a coordinate transformation $x^\mu \to x^{\mu'}$ and a vector V, the vector components transform under change of coordinate basis as

$$V^{\mu'} = \frac{\partial x^{\mu'}}{\partial x^\mu} V^\mu \tag{1}$$

The components of a covector ω transform under change of coordinate basis as

$$\omega_{\mu'} = \frac{\partial x^\mu}{\partial x^{\mu'}} \omega_\mu \tag{2}$$

1.1.1 Inverse Transformations

Show that the transformation matrices of vector (1) and covector (2) are the inverse of each other.

We multiply the matrices

$$\frac{\partial x^{\mu'}}{\partial x^\rho} \frac{\partial x^\rho}{\partial x^{\nu'}} = \frac{\partial x^{\mu'}}{\partial x^{\nu'}} = \delta^{\mu'}_{\nu'} \tag{3}$$

and also in the opposite order

$$\frac{\partial x^\mu}{\partial x^{\rho'}} \frac{\partial x^{\rho'}}{\partial x^\nu} = \frac{\partial x^\mu}{\partial x^\nu} = \delta^\mu_\nu \tag{4}$$

Therefore $\frac{\partial x^{\mu'}}{\partial x^\mu}$ and $\frac{\partial x^\mu}{\partial x^{\mu'}}$ are inverse matrices, and the vector and covector components transform the opposite way. The matrix $J^{\mu'}_{\ \ \mu} = \frac{\partial x^{\mu'}}{\partial x^\mu}$ is the *Jacobian matrix* of the coordinate transformation and $J^\mu_{\ \ \mu'} = \frac{\partial x^\mu}{\partial x^{\mu'}}$ is the inverse Jacobian matrix.

1.1.2 Transforming from Cartesian to spherical coordinates

Find the components in spherical basis of the following covector field , given in Cartesian coordinates as

$$A_\mu = \left(xy, 2y - z^2, xz\right) \tag{5}$$

The coordinates transformation is

$$x = rsin\theta cos\phi \tag{6}$$

$$y = rsin\theta sin\phi \tag{7}$$

$$z = rcos\theta \tag{8}$$

The old coordinates are $x^{\mu} = (x, y, z)$ and the new coordinates are $x^{\mu'} = (r, \theta, \phi)$.

We need to use formula (2). The transformation matrix is

$$J^{\mu}{}_{\mu'} = \frac{\partial x^{\mu}}{\partial x^{\mu'}} = \begin{pmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial \phi} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial \phi} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial \phi} \end{pmatrix} = \begin{pmatrix} sin\theta cos\phi & rcos\theta cos\phi & -rsin\theta sin\phi \\ sin\theta sin\phi & rcos\theta sin\phi & rsin\theta cos\phi \\ cos\theta & -rsin\theta & 0 \end{pmatrix} \tag{9}$$

The components of (5) in spherical coordinates are

$$A_x = xy = r^2 sin^2\theta sin\phi cos\phi \tag{10}$$

$$A_y = 2y - z^2 = 2rsin\theta sin\phi - r^2 cos^2\phi \tag{11}$$

$$A_z = xz = r^2 sin\theta cos\theta cos\phi \tag{12}$$

By (2), the components of the covector field in spherical basis are

$$\begin{aligned} A_r &= \frac{\partial x}{\partial r} A_x + \frac{\partial y}{\partial r} A_y + \frac{\partial z}{\partial r} A_z \\ &= (sin\theta cos\phi)\left(r^2 sin^2\theta sin\phi cos\phi\right) + (sin\theta sin\phi)\left(2rsin\theta sin\phi - r^2 cos^2\phi\right) \\ &+ (cos\theta)\left(r^2 sin\theta cos\theta cos\phi\right) \end{aligned} \tag{13}$$

$$\begin{aligned} A_\theta &= \frac{\partial x}{\partial \theta} A_x + \frac{\partial y}{\partial \theta} A_y + \frac{\partial z}{\partial \theta} A_z \\ &= (rcos\theta cos\phi)\left(r^2 sin^2\theta sin\phi cos\phi\right) + (rcos\theta sin\phi)\left(2rsin\theta sin\phi - r^2 cos^2\phi\right) \\ &+ (-rsin\theta)\left(r^2 sin\theta cos\theta cos\phi\right) \end{aligned} \tag{14}$$

$$\begin{aligned} A_\phi &= \frac{\partial x}{\partial \phi} A_x + \frac{\partial y}{\partial \phi} A_y + \frac{\partial z}{\partial \phi} A_z \\ &= (-rsin\theta sin\phi)\left(r^2 sin^2\theta sin\phi cos\phi\right) + (rsin\theta cos\phi)\left(r^2 sin\theta cos\theta cos\phi\right) \end{aligned} \tag{15}$$

1.1.3 Lorentz Transformation

Show that for Lorentz boost coordinates transformation, the transformation rule (1) leads to the familiar transformation of vector components.

The coordinates transformation of Lorentz boost along the x direction is

$$t' = \gamma t + \gamma v x \tag{16}$$

$$x' = \gamma x + \gamma v t \tag{17}$$

$$y' = y \tag{18}$$

$$z' = z \tag{19}$$

The transformation matrix is

$$J^{\mu'}_{\ \mu} = \frac{\partial x^{\mu'}}{\partial x^{\mu}} = \begin{pmatrix} \frac{\partial t'}{\partial t} & \frac{\partial t'}{\partial x} & \frac{\partial t'}{\partial y} & \frac{\partial t'}{\partial z} \\ \frac{\partial x'}{\partial t} & \frac{\partial x'}{\partial x} & \frac{\partial x'}{\partial y} & \frac{\partial x'}{\partial z} \\ \frac{\partial y'}{\partial t} & \frac{\partial y'}{\partial x} & \frac{\partial y'}{\partial y} & \frac{\partial y'}{\partial z} \\ \frac{\partial z'}{\partial t} & \frac{\partial z'}{\partial x} & \frac{\partial z'}{\partial y} & \frac{\partial z'}{\partial z} \end{pmatrix} = \begin{pmatrix} \gamma & \gamma v & 0 & 0 \\ \gamma v & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \Lambda^{\mu'}_{\ \mu} \tag{20}$$

Therefore the vector components transform as

$$V^{\mu'} = \Lambda^{\mu'}_{\ \mu} V^{\mu} \tag{21}$$

Remark: Any **linear** coordinate transformation can be written in short matrix notation as $x' = Ax$, where A is a constant matrix. The Jacobian matrix is just $J = \frac{\partial x'}{\partial x} = A$, i.e., the constant coefficients of the coordinates transformation. Only in this special case the coordinates themselves transform as a vector. Lorentz transformation matrix also has determinant 1.

1.2 Tensor Transformation

1.2.1 Transformation of the metric tensor components

Find the transformation rule for the components of the metric tensor from the invariance of the line element.

$$\begin{aligned} ds^2 &= g_{\mu'\nu'} dx^{\mu'} dx^{\nu'} = g_{\mu\nu} dx^{\mu} dx^{\nu} \\ &= g_{\mu\nu} \frac{\partial x^{\mu}}{\partial x^{\mu'}} \frac{\partial x^{\nu}}{\partial x^{\nu'}} dx^{\mu'} dx^{\nu'} \end{aligned} \tag{22}$$

Therefore

$$g_{\mu'\nu'} = \frac{\partial x^{\mu}}{\partial x^{\mu'}}\frac{\partial x^{\nu}}{\partial x^{\nu'}} g_{\mu\nu} \tag{23}$$

Transformation of the metric components

This is of course with agreement with the transformation of a $\begin{pmatrix} 0 \\ 2 \end{pmatrix}$ tensor components.

1.2.2 Invariant volume element

Show that $\sqrt{g} d^d x$ is an invariant volume element, where $g \equiv det\left(g_{\mu\nu}\right)$ and $d^d x = det\left(dx^{\mu}\right)$. d is the dimension of space.

The metric determinant transforms as

$$g' = det\left(g_{\mu'\nu'}\right) = det\left(\frac{\partial x^{\mu}}{\partial x^{\mu'}}\frac{\partial x^{\nu}}{\partial x^{\nu'}} g_{\mu\nu}\right) = det\left(\frac{\partial x^{\mu}}{\partial x^{\mu'}}\right) det\left(\frac{\partial x^{\nu}}{\partial x^{\nu'}}\right) det\left(g_{\mu\nu}\right)$$
$$= \left[det\left(\frac{\partial x}{\partial x'}\right)\right]^2 g \tag{24}$$

Notice that it is not a scalar. $d^d x$ is also not a scalar, it transforms as

$$d^d x' = det\left(dx^{\mu'}\right) = det\left(\frac{\partial x^{\mu'}}{\partial x^{\mu}} dx^{\mu}\right) = det\left(\frac{\partial x^{\mu'}}{\partial x^{\mu}}\right) det\left(dx^{\mu}\right)$$
$$= det\left(\frac{\partial x'}{\partial x}\right) d^d x \tag{25}$$

This is the famous Jacobian determinant used when changing variables in integration.

Therefore

$$\sqrt{g}' d^d x' = \sqrt{\left[det\left(\frac{\partial x}{\partial x'}\right)\right]^2 g}\left(det\left(\frac{\partial x'}{\partial x}\right)\right) d^d x = det\left(\frac{\partial x}{\partial x'}\right) det\left(\frac{\partial x'}{\partial x}\right)\sqrt{g} d^d x = \sqrt{g} d^d x \tag{26}$$

is invariant. We used the fact that $\frac{\partial x}{\partial x'} = \left(\frac{\partial x'}{\partial x}\right)^{-1}$ and the determinant of an inverse matrix is the inverse of the determinant $det\left(A^{-1}\right) = \left(det\left(A\right)\right)^{-1}$.

1.2.3 Contraction is basis independent

For example, take a tensor $T^{\mu}{}_{\nu\rho}$ and show that the contraction $T^{\mu}{}_{\mu\rho}$ is basis independent, i.e., that $T^{\mu'}{}_{\mu'\rho} = T^{\mu}{}_{\mu\rho}$.

$$T^{\mu'}{}_{\mu'\rho} = \frac{\partial x^{\mu'}}{\partial x^{\sigma}} \frac{\partial x^{\nu}}{\partial x^{\mu'}} T^{\sigma}{}_{\nu\rho} = \delta^{\nu}_{\sigma} T^{\sigma}{}_{\nu\rho} = T^{\sigma}{}_{\sigma\rho} \tag{27}$$

Thus, the "contracted part" is a scalar, and overall $T^{\mu}{}_{\mu\rho}$ is a covector.

2 Covariant Derivative

The ν component of the covariant derivative of a vector field V in the direction of e_μ is

$$\nabla_\mu V^\nu = \frac{\partial V^\nu}{\partial x^\mu} + \Gamma^{\nu}_{\mu\rho} V^\rho \tag{28}$$

The ν component of the covariant derivative of a covector field ω in the direction of e_μ is

$$\nabla_\mu \omega_\nu = \frac{\partial \omega_\nu}{\partial x^\mu} - \Gamma^{\rho}_{\mu\nu} \omega_\rho \tag{29}$$

For the covariant derivative of a tensor field we add or subtract Christoffels for each index, plus for upper index and minus for lower index. For example,

$$\nabla_\mu T^{\nu}{}_{\rho} = \frac{\partial T^{\nu}{}_{\rho}}{\partial x^\mu} + \Gamma^{\nu}_{\mu\sigma} T^{\sigma}{}_{\rho} - \Gamma^{\sigma}_{\mu\rho} T^{\nu}{}_{\sigma} \tag{30}$$

$$\nabla_\mu T_{\nu\rho} = \frac{\partial T_{\nu\rho}}{\partial x^\mu} - \Gamma^{\sigma}_{\mu\nu} T_{\sigma\rho} - \Gamma^{\sigma}_{\mu\rho} T_{\nu\sigma} \tag{31}$$

$$\nabla_\mu T^{\nu\rho} = \frac{\partial T^{\nu\rho}}{\partial x^\mu} + \Gamma^{\nu}_{\mu\sigma} T^{\sigma\rho} + \Gamma^{\rho}_{\mu\sigma} T^{\nu\sigma} \tag{32}$$

2.1 Covariant Differentiation on a Sphere

The metric of a sphere in (θ, ϕ) coordinates is

$$ds^2 = a^2 \left(d\theta^2 + sin^2\theta d\phi^2 \right) \tag{33}$$

1. Calculate the four components of the tensor $\nabla_i V^j$, for the vector field V with components $V^i = (0, 1)$.
2. Calculate $\nabla_\theta \nabla_\phi V^\theta$ and $\nabla_\phi \nabla_\theta V^\theta$. Does the covariant derivatives commute?

2.1.1 First derivatives

We know the Christoffel symbols of the sphere. The non-vanishing ones are

$$\Gamma^{\theta}_{\phi\phi} = -sin\theta cos\theta \tag{34}$$

$$\Gamma^{\phi}_{\phi\theta} = \frac{cos\theta}{sin\theta} \tag{35}$$

The covariant derivative is

$$\nabla_i V^j = \frac{\partial V^j}{\partial x^i} + \Gamma^j_{ik} V^k = \Gamma^j_{ik} V^k \tag{36}$$

since $\frac{\partial V^j}{\partial x^i} = 0$. The four components are:

$$\nabla_\theta V^\theta = \Gamma^\theta_{\theta k} V^k = 0 \tag{37}$$

since all $\Gamma^\theta_{\theta k} = 0$.

$$\nabla_\theta V^\phi = \Gamma^\phi_{\theta k} V^k = \Gamma^\phi_{\theta\phi} V^\phi = \frac{cos\theta}{sin\theta} \tag{38}$$

since $\Gamma^\phi_{\theta k} \neq 0$ only for $k = \phi$ and $V^\phi = 1$.

$$\nabla_\phi V^\theta = \Gamma^\theta_{\phi k} V^k = \Gamma^\theta_{\phi\phi} V^\phi = -sin\theta cos\theta \tag{39}$$

since $\Gamma^\theta_{\phi k} \neq 0$ only for $k = \phi$ and $V^\phi = 1$.

$$\nabla_\phi V^\phi = \Gamma^\phi_{\phi k} V^k = \Gamma^\phi_{\phi\theta} V^\theta = 0 \tag{40}$$

since $\Gamma^\phi_{\phi k} \neq 0$ only for $k = \theta$ and $V^\theta = 0$.

2.1.2 Second derivatives

$$\begin{aligned}
\nabla_\theta \nabla_\phi V^\theta &= \frac{\partial}{\partial\theta}\left(\nabla_\phi V^\theta\right) + \Gamma^\theta_{\theta k} \nabla_\phi V^k - \Gamma^k_{\theta\phi} \nabla_k V^\theta \\
&= \frac{\partial}{\partial\theta}\left(-sin\theta cos\theta\right) + 0 - \Gamma^\phi_{\theta\phi} \nabla_\phi V^\theta \\
&= -cos^2\theta + sin^2\theta - \frac{cos\theta}{sin\theta}\left(-sin\theta cos\theta\right) = sin^2\theta
\end{aligned} \tag{41}$$

$$
\begin{aligned}
\nabla_\phi \nabla_\theta V^\theta = & \frac{\partial}{\partial \phi}\left(\nabla_\theta V^\theta\right) + \Gamma^\theta_{\phi k} \nabla_\theta V^k - \Gamma^k_{\phi\theta} \nabla_k V^\theta \\
= & \Gamma^\theta_{\phi\phi} \nabla_\theta V^\phi - \Gamma^\phi_{\phi\theta} \nabla_\phi V^\theta \\
= & -sin\theta cos\theta \frac{cos\theta}{sin\theta} - \frac{cos\theta}{sin\theta}\left(-sin\theta cos\theta\right) = 0
\end{aligned} \tag{42}
$$

So,

$$
\left(\nabla_\theta \nabla_\phi - \nabla_\phi \nabla_\theta\right) V^\theta \neq 0 \tag{43}
$$

2.2 Metric Compatibility

The "straightest lines", i.e., trajectories of zero covariant acceleration at any point along the curve, are determined by the covariant derivative. They satisfy the equation

$$
\frac{d^2 x^\rho}{d\tau^2} + \Gamma^\rho_{\mu\nu} \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau} = 0 \tag{44}
$$

The "shortest lines", i.e., curves that extermizes the length functional, are determined by the metric. They satisfy the equation

$$
\frac{d^2 x^\rho}{d\tau^2} + \left[\frac{1}{2} g^{\rho\sigma}\left(\partial_\mu g_{\nu\sigma} + \partial_\nu g_{\mu\sigma} - \partial_\sigma g_{\mu\nu}\right)\right] \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau} = 0 \tag{45}
$$

Demanding that the "straightest lines" and the "shortest lines" coincide, means that (44) and (45) are the same equation, called the *geodesic equation.* It happens when the Christoffels are given by the metric as

$$
\Gamma^\rho_{\mu\nu} = \frac{1}{2} g^{\rho\sigma}\left(\partial_\mu g_{\nu\sigma} + \partial_\nu g_{\mu\sigma} - \partial_\sigma g_{\mu\nu}\right) \tag{46}
$$

In this case we say that the covariant derivative is compatible with the metric.

Show that from (46) it follows that the metric is *covariantly constant*

$$
\nabla_\rho g_{\mu\nu} = 0 \tag{47}
$$

Notice that (47) is a third rank tensor equation, so it is zero in any basis. Equation (47) is called *metric compatibility.*

$$
\begin{aligned}
\nabla_\rho g_{\mu\nu} = &\partial_\rho g_{\mu\nu} - \Gamma^\sigma_{\rho\mu} g_{\sigma\nu} - \Gamma^\sigma_{\rho\nu} g_{\mu\sigma} \\
=&\partial_\rho g_{\mu\nu} - \frac{1}{2} g^{\sigma\lambda} \left(\partial_\rho g_{\mu\lambda} + \partial_\mu g_{\rho\lambda} - \partial_\lambda g_{\rho\mu}\right) g_{\sigma\nu} - \frac{1}{2} g^{\sigma\lambda} \left(\partial_\rho g_{\nu\lambda} + \partial_\nu g_{\rho\lambda} - \partial_\lambda g_{\rho\nu}\right) g_{\mu\sigma} \\
=&\partial_\rho g_{\mu\nu} - \frac{1}{2} \delta^\lambda_\nu \left(\partial_\rho g_{\mu\lambda} + \partial_\mu g_{\rho\lambda} - \partial_\lambda g_{\rho\mu}\right) - \frac{1}{2} \delta^\lambda_\mu \left(\partial_\rho g_{\nu\lambda} + \partial_\nu g_{\rho\lambda} - \partial_\lambda g_{\rho\nu}\right) \\
=&\partial_\rho g_{\mu\nu} - \frac{1}{2} \left(\partial_\rho g_{\mu\nu} + \partial_\mu g_{\rho\nu} - \partial_\nu g_{\rho\mu}\right) - \frac{1}{2} \left(\partial_\rho g_{\nu\mu} + \partial_\nu g_{\rho\mu} - \partial_\mu g_{\rho\nu}\right) \\
=&0
\end{aligned} \tag{48}
$$

Geometrically, it follows that the dot product (angle) of two vectors that are parallely transported along a curve remains the same.

2.3 Gradient, Divergence and Laplacian

1. Write down the components of gradient of a function f as a contravariant vector $\nabla^\mu f$.
2. Show that

$$
\Gamma^\rho_{\rho\mu} = \frac{1}{\sqrt{g}} \partial_\mu \sqrt{g} \tag{49}
$$

3. Show that the (covariant) divergence of a vector field V is given by

$$
divV = \nabla_\mu V^\mu = \frac{1}{\sqrt{g}} \partial_\mu \left(\sqrt{g} V^\mu\right) \tag{50}
$$

4. Show that the (covariant) Laplacian of a function f is given by

$$
\nabla^2 f \equiv \nabla_\mu \nabla^\mu f = \frac{1}{\sqrt{g}} \partial_\mu \left(\sqrt{g} g^{\mu\nu} \partial_\nu f\right) \tag{51}
$$

2.3.1 Gradient

The covariant derivative acts on a function like a partial derivative.

$$
\nabla_\mu f = \partial_\mu f \tag{52}
$$

Raise the index with the metric

$$
\nabla^\mu f = g^{\mu\nu} \partial_\nu f \tag{53}
$$

Contravariant gradient

2.3.2 Contracted Christoffel

Start with the left hand side of (49).

$$\Gamma^{\rho}_{\rho\mu} = \frac{1}{2} g^{\rho\sigma} \left(\partial_{\rho} g_{\mu\sigma} + \partial_{\mu} g_{\rho\sigma} - \partial_{\sigma} g_{\rho\mu} \right) = \frac{1}{2} g^{\rho\sigma} \partial_{\mu} g_{\rho\sigma} \tag{54}$$

The first and last term are the same and cancel, since both are contracted as $g^{\alpha\beta}\partial_{\alpha} g_{\beta\mu}$ (the metric is symmetric).

To unravel the right hand side of (49) we need to differentiate the metric determinant g. Recall that for ordinary numbers the logarithm function satisfies that the logarithm of a product equals the sum of the logs, i.e.,

$$ln\left(ab\right) = ln\left(a\right) + ln\left(b\right) \tag{55}$$

A generalization exists for matrices, where the product in (55) is replaced by the determinant of the matrix (the product of its eigenvalues), and the sum in (55) is replaced by the trace of the matrix (the sum of its eigenvalues). For a matrix A

$$ln\left(detA\right) = Tr\left(lnA\right) \tag{56}$$

Differentiate

$$\frac{1}{detA}\partial_{\mu}\left(detA\right) = Tr\left(A^{-1}\partial_{\mu}A\right) \tag{57}$$

For the metric $g_{\mu\nu}$, its inverse $g^{\mu\nu}$, and its determinant g we have

$$\frac{1}{g}\partial_{\mu} g = g^{\rho\sigma}\partial_{\mu} g_{\rho\sigma} \tag{58}$$

where we used one contraction as the matrix multiplication and second contraction as the trace.

Now with the square root

$$\frac{1}{\sqrt{g}}\partial_{\mu}\sqrt{g} = \frac{1}{\sqrt{g}}\frac{1}{2\sqrt{g}}\partial_{\mu} g = \frac{1}{2}\frac{1}{g}\partial_{\mu} g = \frac{1}{2} g^{\rho\sigma}\partial_{\mu} g_{\rho\sigma} \tag{59}$$

Since (54) and (59) equal the same thing, we have

$$\Gamma^{\rho}_{\rho\mu} = \frac{1}{\sqrt{g}}\partial_{\mu}\sqrt{g} \tag{60}$$

Contracted Christoffel

2.3.3 Divergence

Now we can use (60) to write the divergence.

$$\begin{aligned}\nabla_\mu V^\mu &= \partial_\mu V^\mu + \Gamma^\mu_{\mu\nu} V^\nu \\ &= \partial_\mu V^\mu + \frac{1}{\sqrt{g}} \left(\partial_\nu \sqrt{g}\right) V^\nu = \partial_\mu V^\mu + \frac{1}{\sqrt{g}} \left(\partial_\mu \sqrt{g}\right) V^\mu\end{aligned} \tag{61}$$

By the product rule for derivative we get

Divergence of a vector field

$$\nabla_\mu V^\mu = \frac{1}{\sqrt{g}} \partial_\mu \left(\sqrt{g} V^\mu\right) \tag{62}$$

2.3.4 Laplacian

The Laplacian is just the divergence of the gradient. Use (53) and (62)

$$\nabla_\mu \nabla^\mu f = \nabla_\mu \left(g^{\mu\nu} \partial_\nu f\right) = \frac{1}{\sqrt{g}} \partial_\mu \left(\sqrt{g} g^{\mu\nu} \partial_\nu f\right) \tag{63}$$

Laplacian of a function

$$\nabla_\mu \nabla^\mu f = \frac{1}{\sqrt{g}} \partial_\mu \left(\sqrt{g} g^{\mu\nu} \partial_\nu f\right) \tag{64}$$

Notice that in flat space in Cartesian coordinates, where $g_{ij} = \delta_{ij}$ and $\Gamma^k_{ij} = 0$ (so the covariant derivative reduces to the partial derivative), both sides of (53),(62) and (64) reduce to the familiar standard expressions.

2.4 Killing's Equation

Suppose the metric does not depend on x^1. Then, there is a Killing vector field

$$\xi^\mu = (0, 1, 0, 0) \tag{65}$$

Show that

Killing's equation

$$\nabla_\mu \xi_\nu + \nabla_\nu \xi_\mu = 0 \tag{66}$$

Notice that in (66), ξ is the dual vector field, it has an index down, as opposed to the Killing vector field itself (65). We can read (66) as "the symmetric part of the second rank tensor $\nabla_\mu \xi_\nu$ vanish". (66) is called *Killing's equation*. Since it is a tensorial equation, it has the same form in any basis (in other coordinates, where ξ has more complicated components). The general way to find all the Killing vector fields is to find all the solutions of Killing's equation.

$$\begin{aligned}
\nabla_\mu \xi_\nu + \nabla_\nu \xi_\mu = & \partial_\mu \xi_\nu - \Gamma^\rho_{\mu\nu}\xi_\rho + \partial_\nu \xi_\mu - \Gamma^\rho_{\nu\mu}\xi_\rho \\
= & \partial_\mu \left(g_{\nu\rho}\xi^\rho\right) + \partial_\nu \left(g_{\mu\rho}\xi^\rho\right) - 2\Gamma^\rho_{\mu\nu} g_{\rho\sigma}\xi^\sigma \\
= & \partial_\mu g_{\nu 1} + \partial_\nu g_{\mu 1} - g_{\rho 1} g^{\rho\lambda}\left(\partial_\mu g_{\nu\lambda} + \partial_\nu g_{\mu\lambda} - \partial_\lambda g_{\mu\nu}\right) \\
= & \partial_\mu g_{\nu 1} + \partial_\nu g_{\mu 1} - \delta^\lambda_1 \left(\partial_\mu g_{\nu\lambda} + \partial_\nu g_{\mu\lambda} - \partial_\lambda g_{\mu\nu}\right) \\
= & \partial_\mu g_{\nu 1} + \partial_\nu g_{\mu 1} - \left(\partial_\mu g_{\nu 1} + \partial_\nu g_{\mu 1} - \partial_1 g_{\mu\nu}\right) \\
= & \partial_1 g_{\mu\nu} = 0
\end{aligned} \tag{67}$$

where we used

$$g_{\nu\sigma}\xi^\sigma = g_{\nu 1} \tag{68}$$

and

$$\partial_1 g_{\mu\nu} = 0 \tag{69}$$