The forward recurrence time

Now we turn to investigate the PDF of the forward recurrence time $f_E(t,E)$. Namely, the PDF of the time we have to wait for the event if we started our measurement at time t and the process itself started at time 0. We define $f_{E,n}(t,E)$ as the joint probability of finding the forward recurrence time with value E and n events in the time interval t. Once we find $f_{E,n}(t,E)$ we can sum over n to find $f_E(t,E)$. $E = t_{n+1} - t$ and $t_n < t < t_{n+1}$ and therefore,

$$f_{E,n}(t,E) = \langle \delta(t_{n+1} - t - E) I(t_n < t < t_{n+1}) \rangle . \tag{1}$$

We will use the double Laplace Transform

$$\widehat{f}_{E,n}(s,u) \equiv \int_0^\infty dt e^{-st} \int_0^\infty dE e^{-uE} f_{E,n}(t,E) . \tag{2}$$

We first take the integral with respect to E

$$\begin{aligned} \widehat{f}_{E,n}(s,u) &\equiv \left\langle \int_0^\infty dt e^{-st} \int_0^\infty dE e^{-uE} \delta(t_{n+1} - t - E) I(t_n < t < t_{n+1}) \right\rangle \\ &= \left\langle \int_0^\infty dt e^{-st} e^{-u(t_{n+1}-t)} I(t_n < t < t_{n+1}) \right\rangle \\ &= \left\langle \frac{e^{-st_n} e^{-u(t_{n+1}-t_n)} - e^{-ut_{n+1}} e^{-(s-u)t_{n+1}}}{s-u} \right\rangle \\ &= \left[\widehat{\psi}(s)\right]^n \frac{\widehat{\psi}(u) - \widehat{\psi}(s)}{s-u} . \end{aligned} \tag{3}$$

Summing over n we obtain the double Laplace transform, $t \to s$ and $E \to u$ of $f_E(t,E)$,

$$\widehat{f}_E(s,u) = \frac{\widehat{\psi}(u) - \widehat{\psi}(s)}{s-u} \frac{1}{1 - \widehat{\psi}(s)} . \tag{4}$$

Note that generally the PDF depends on the age of the process t since its Laplace transform depends on s.

We first consider the case of a finite first moment, *case* 2 above. In this case, the small s limit yields

$$\widehat{f}_E(s,u) \sim \frac{1 - \widehat{\psi}(u)}{u} \frac{1}{\langle \tau \rangle s} . \tag{5}$$

Inverting the Laplace transform with respect to s we find

$$\widehat{f}_E(t,u) \sim \frac{1 - \widehat{\psi}(u)}{u} \frac{1}{\langle \tau \rangle} . \tag{6}$$

Earlier we showed that $\frac{1-\widehat{\psi}(u)}{u}$ is the Laplace transform of the probability that an event has not occurred up to time E (see equation (**??**)). Inverting the Laplace transform with respect to u we find

$$f_E(t,E) \sim \frac{R(E)}{\langle \tau \rangle} = \frac{1}{\langle \tau \rangle} \left(1 - \int_0^E \psi(\tau) d\tau \right) . \tag{7}$$

Note that this result is independent of t and therefore we may denote it as ${}^{eq}f_E(E)$. A very different behavior is obtained for *case* 1 in which the first moment diverges. Taking the limit of $s, u \to 0$ keeping the ratio s/u arbitrary we may write equation (4) as

$$\widehat{f}_E(s,u) \sim \frac{s^\alpha - u^\alpha}{s-u} \frac{1}{s^\alpha} . \tag{8}$$

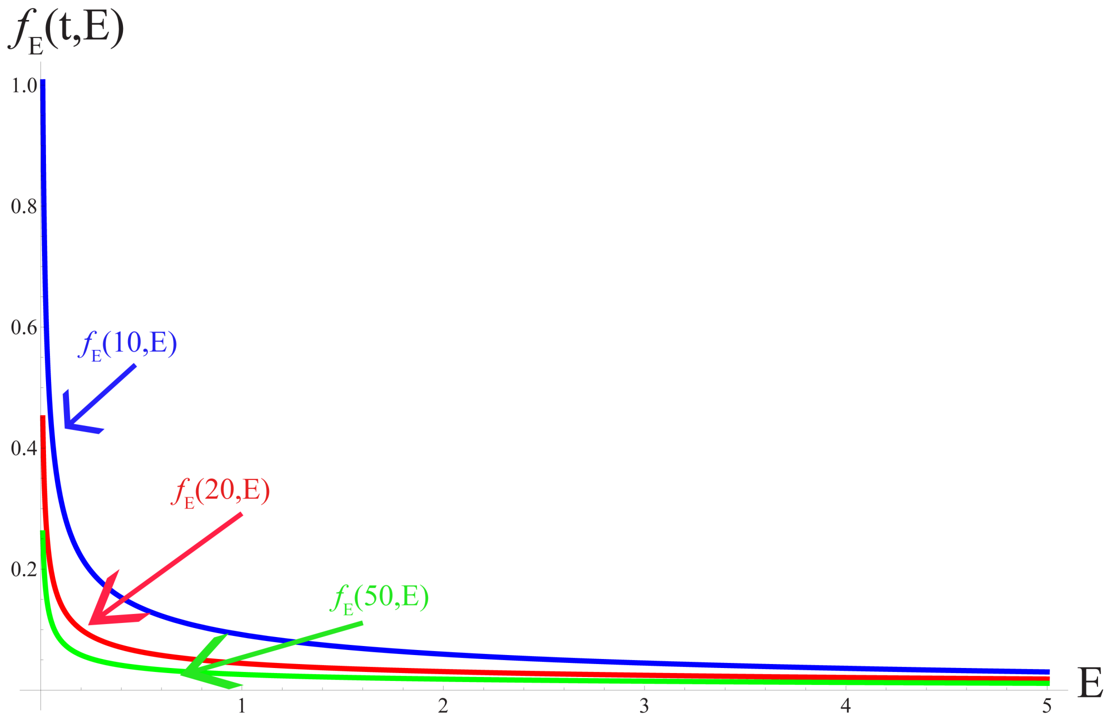

Figure 1: Plot of $f_E(t, E)$ with $\alpha = 1/2$. The different curves correspond to different times.

Inverting the double Laplace transform (details are omitted here but can be provided if you are interested) we obtain

$$f_E(t, E) \sim \frac{\sin(\pi\alpha)}{\pi} \frac{t^\alpha}{E^\alpha (t + E)}. \tag{9}$$

This distribution was first derived by Dynkin. It shows that as the system gets older, i.e., t is larger, the PDF of E becomes more stretched.Please derive at home the PDF of the backward recurrence time. It should be easy if you follow the steps of the derivation of the forward recurrence time PDF.

Statistical Aging

So far we only considered a renewal process in which our measurement starts when an event occurred, namely $t_1 = \tau_1$ and τ_1 is characterized by the same PDF as the other waiting times between events. In reality, this is not necessarily the case. In what follows we consider a process starting at time $-t_a$, while our measurement starts at time $t = 0$. We will find the probability of n events in the interval $[0, t]$ given that our process started at time $-t_a$, $W(n, t; t_a)$. Note, that in this case, all the waiting times statistics is the same and independent of t_a with the exception of τ_1. However, τ_1 is just the forward recurrence time whose statistics was obtained above. We denote its PDF as $h(\tau; t_a)$. According to our derivation for the forward recurrence time, the double Laplace transform of $h(\tau; t_a)$, $t_a \to u$ and $\tau \to s$ is

$$\widehat{h}(s; u) = \frac{\widehat{\psi}(s) - \widehat{\psi}(u)}{u - s} \frac{1}{1 - \widehat{\psi}(u)}. \tag{10}$$

Repeating a similar procedure to the one we used in our derivation of $\widehat{W}(n, s)$ we find

$$\widehat{W}(n, s; u) = \begin{cases} \frac{1 - u\widehat{h}(s;u)}{su} & n = 0 \\ \widehat{h}(s; u)\, \widehat{\psi}^{n-1}(s)\, \frac{1 - \widehat{\psi}(s)}{s} & n \geq 1 \end{cases} \tag{11}$$

(The derivations for the two cases are below

$$\widehat{W}(0,s;u) = \left\langle \int_0^\infty e^{-ut_a} dt_a \int_0^\infty e^{-st} I(0<t<t_1)\, dt \right\rangle \tag{12}$$
$$= \left\langle \int_0^\infty e^{-ut_a} dt_a \int_0^{t_1} e^{-st} dt \right\rangle$$
$$= \left\langle \int_0^\infty e^{-ut_a} dt_a \frac{1-e^{-s\tau_1}}{s} \right\rangle$$

$$= \frac{1}{su} - \frac{1}{s} \left\langle \int_0^\infty e^{-ut_a} e^{-s\tau_1} dt_a \right\rangle$$
$$= \frac{1}{su} - \frac{1}{s} \int_0^\infty \int_0^\infty e^{-s\tau_1} e^{-ut_a} h(\tau_1;t_a)\, dt_a d\tau_1$$
$$= \frac{1}{su} - \frac{\widehat{h}(s;u)}{s}$$

and

$$\widehat{W}(n,s;u) = \left\langle \int_0^\infty e^{-ut_a} dt_a \int_0^\infty e^{-st} I(t_n<t<t_{n+1})\, dt \right\rangle \tag{13}$$
$$= \left\langle \int_0^\infty e^{-ut_a} dt_a \int_{t_n}^{t_{n+1}} e^{-st} dt \right\rangle$$

$$= \left\langle \int_0^\infty e^{-ut_a} dt_a \frac{e^{-st_n} - e^{-st_{n+1}}}{s} \right\rangle$$
$$= \left\langle \int_0^\infty e^{-ut_a} e^{-s\tau_1} dt_a \right\rangle \left[\widehat{\psi}(s)\right]^{n-1} \frac{1-\widehat{\psi}(s)}{s}$$
$$= \widehat{h}(s;u) \left[\widehat{\psi}(s)\right]^{n-1} \frac{1-\widehat{\psi}(s)}{s}$$

).

The double Laplace transform of the average number of events in the interval $[0,t]$ is

$$\langle \widehat{n}(s;u) \rangle = \sum_{n=0}^{\infty} n \widehat{W}(n,s;u) = \widehat{h}(s;u) \frac{1-\widehat{\psi}(s)}{s} \sum_{n=1}^{\infty} n \left[\widehat{\psi}(s)\right]^{n-1} \tag{14}$$
$$= \frac{\widehat{h}(s;u)}{s\left(1-\widehat{\psi}(s)\right)}.$$

To consider the equilibrium situation we take the limit $t_a \to \infty$. If the average waiting time is finite, we found (see equation (10))

$$^{eq}\widehat{h}(s) \equiv \lim_{t_a\to\infty} \widehat{h}_{t_a}(s) = \frac{1-\widehat{\psi}(s)}{\langle \tau \rangle s}. \tag{15}$$

(In the double Laplace space, we find that in the limit $u \to 0$

$$\widehat{h}(s;u) = \frac{1-\widehat{\psi}(s)}{s}\frac{1}{\langle\tau\rangle u}.$$

Inverting the Laplace transform with respect to u we find the expression in eq. (15)).

For such a case, the process in the interval $[0,t]$ is called an equilibrium renewal process. Note that for this process, the statistics of the number of events is independent of the age t_a and $W(n,t;t_a) \to {}^{eq}W(n,t)$. Inserting the above expression into equation (11) we find

$$\begin{aligned} {}^{eq}\widehat{W}(n\geq 1,s) &= \widehat{\psi}^{n-1}(s)\frac{\left[1-\widehat{\psi}(s)\right]^2}{\langle\tau\rangle s^2}, \\ {}^{eq}\widehat{W}(n=0,s) &= \frac{1}{s}-\frac{1-\widehat{\psi}(s)}{\langle\tau\rangle s^2}, \end{aligned} \tag{16}$$

where we used,

$$\begin{aligned} {}^{eq}\widehat{W}(0,s) &= \left\langle\int_0^\infty e^{-st}I(0<t<t_1)\,dt\right\rangle \\ &= \left\langle\int_0^{t_1} e^{-st}dt\right\rangle \\ &= \left\langle\frac{1-e^{-s\tau_1}}{s}\right\rangle \\ &= \frac{1-{}^{eq}\widehat{h}(s)}{s} \end{aligned} \tag{17}$$

and

$$\begin{aligned} {}^{eq}\widehat{W}(n\geq 1,s) &= \left\langle\int_0^\infty e^{-st}I(t_n<t<t_{n+1})\,dt\right\rangle \\ &= \left\langle\int_{t_n}^{t_{n+1}} e^{-st}dt\right\rangle \\ &= \left\langle\frac{e^{-st_n}-e^{-st_{n+1}}}{s}\right\rangle \\ &= \left\langle e^{-s\tau_1}\right\rangle\left[\widehat{\psi}(s)\right]^{n-1}\frac{1-\widehat{\psi}(s)}{s} \\ &= {}^{eq}\widehat{h}(s)\left[\widehat{\psi}(s)\right]^{n-1}\frac{1-\widehat{\psi}(s)}{s}. \end{aligned} \tag{18}$$

The average number of events is:

$$\langle\widehat{n}(s;u)\rangle = \frac{\widehat{h}(s;u)}{s\left(1-\widehat{\psi}(s)\right)} = \frac{\frac{\widehat{\psi}(s)-\widehat{\psi}(u)}{u-s}\frac{1}{1-\widehat{\psi}(u)}}{s\left(1-\widehat{\psi}(s)\right)} \tag{19}$$

In order to study the limit of long t_a and t we may expand $\widehat{\psi}(s)$. For the case of finite average waiting time we obtain

$$\langle\widehat{n}(s;u)\rangle \sim \frac{1}{us^2\langle\tau\rangle} \to \langle n(t;t_a)\rangle \sim \frac{t}{\langle\tau\rangle}. \tag{20}$$

Note that the long time limit of the average number is independent of the age of the system.

For the case of diverging average waiting time we may expand $\widehat{\psi}(s) \sim 1 - As^{\alpha}$

$$\langle \widehat{n}(s;u) \rangle \sim \frac{\frac{(u^{\alpha}-s^{\alpha})}{u-s}\frac{1}{u^{\alpha}}}{As^{1+\alpha}} = \frac{1}{Aus^{1+\alpha}}\frac{1-\left(\frac{s}{u}\right)^{\alpha}}{1-s/u} \tag{21}$$

$$\langle n(t;t_a) \rangle \sim \frac{(t+t_a)^{\alpha} - t_a^{\alpha}}{A\Gamma(1+\alpha)}. \tag{22}$$

The result above is easily understood. Remembering that for the case of diverging first moment of the waiting time we found (for a process starting at $t = 0$ right after an event occurred) that $\langle n(t) \rangle \sim \frac{t^{\alpha}}{A\Gamma(1+\alpha)}$, we see that the average number of events is just the difference between the average number of events in an interval of duration $t + t_a$ and the average number of events in an interval of duration t_a.