

Gravity 1 - Recitation 11

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1 Linearized Gravity

1.1 The Formulae

Finding exact solutions to Einstein's equations is hard. Nonetheless, if we have such exact solution, we can make a small perturbation around it and approximate the equations to linear order in the perturbation and find an approximate solution. We will make a perturbation around flat space, i.e., the Minkowski metric solution.

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} \quad (1)$$

where $\eta_{\mu\nu}$ is Minkowski metric in inertial coordinates, and $h_{\mu\nu}(x)$ is the *metric perturbation*, $|h_{\mu\nu}| \ll 1$.

The inverse metric, to first order in h , is

$$g^{\mu\nu} = \eta^{\mu\nu} - h^{\mu\nu} \quad (2)$$

where $h^{\mu\nu} \equiv \eta^{\mu\rho}\eta^{\nu\sigma}h_{\rho\sigma}$. All the raising, lowering, trace, manipulations to first order are done with η .

The Christoffel symbols to first order in h is

$$(\Gamma_{\mu\nu}^{\rho})^{(1)} = \frac{1}{2}\eta^{\rho\sigma}(\partial_{\mu}h_{\nu\sigma} + \partial_{\nu}h_{\mu\sigma} - \partial_{\sigma}h_{\mu\nu}) \quad (3)$$

The Riemann tensor to first order in h is

$$(R_{\rho\sigma\mu\nu})^{(1)} = \frac{1}{2}(\partial_{\mu}\partial_{\sigma}h_{\rho\nu} - \partial_{\nu}\partial_{\sigma}h_{\rho\mu} - \partial_{\mu}\partial_{\rho}h_{\sigma\nu} + \partial_{\nu}\partial_{\rho}h_{\sigma\mu}) \quad (4)$$

The Ricci tensor to first order in h is

$$\begin{aligned} (R_{\mu\nu})^{(1)} &= \frac{1}{2}(\partial_{\mu}\partial_{\rho}h^{\rho}_{\nu} + \partial_{\nu}\partial_{\rho}h^{\rho}_{\mu} - \partial_{\mu}\partial_{\nu}h - \square h_{\mu\nu}) \\ &= \frac{1}{2}(\partial_{\mu}V_{\nu} + \partial_{\nu}V_{\mu} - \square h_{\mu\nu}) \end{aligned} \quad (5)$$

where

$$V_{\mu} \equiv \partial_{\nu}h^{\nu}_{\mu} - \frac{1}{2}\partial_{\mu}h \quad (6)$$

and h is the **trace** of $h_{\mu\nu}$

$$h \equiv \eta^{\mu\nu}h_{\mu\nu} = h^{\nu}_{\nu} \quad (7)$$

and \square to zero order in h is

$$\square = \eta^{\rho\sigma} \partial_\rho \partial_\sigma = -\partial_t^2 + \partial_x^2 + \partial_y^2 + \partial_z^2 \quad (8)$$

The Ricci scalar to first order in h is

$$R^{(1)} = \partial_\rho \partial_\sigma h^{\rho\sigma} - \square h \quad (9)$$

The Einstein tensor to first order in h is

$$(G_{\mu\nu})^{(1)} = \frac{1}{2} (\partial_\mu \partial_\sigma h_{\rho\nu} - \partial_\nu \partial_\sigma h_{\rho\mu} - \partial_\mu \partial_\rho h_{\sigma\nu} + \partial_\nu \partial_\rho h_{\sigma\mu} - \eta_{\mu\nu} \partial_\rho \partial_\sigma h^{\rho\sigma} + \eta_{\mu\nu} \square h) \quad (10)$$

Gauge transformation of $h_{\mu\nu}$ is

$$h'_{\mu\nu} = h_{\mu\nu} - \partial_\mu \xi_\nu - \partial_\nu \xi_\mu \quad (11)$$

The Lorenz/harmonic gauge is

$$V_\mu \equiv \partial_\nu h^\nu{}_\mu - \frac{1}{2} \partial_\mu h = 0 \quad (12)$$

It is equivalent to a choice of *harmonic coordinates*, i.e., **coordinate functions** with zero Laplacian

$$\square x^\mu = 0 \quad (13)$$

The linearized vacuum Einstein equation is

$$(R_{\mu\nu})^{(1)} = 0 \quad (14)$$

In the Lorenz gauge it is the wave equation

$$\square h_{\mu\nu} = 0 \quad (15)$$

1.2 Gauge Invariance

Exercise: Show explicitly that the linearized Riemann tensor (4) is invariant under gauge transformation (11).

Plug (11) into (4)

$$\begin{aligned}
(R'_{\rho\sigma\mu\nu})^{(1)} &= \frac{1}{2} (\partial_\mu \partial_\sigma (h_{\rho\nu} - \partial_\rho \xi_\nu - \partial_\nu \xi_\rho) - \partial_\nu \partial_\sigma (h_{\rho\mu} - \partial_\rho \xi_\mu - \partial_\mu \xi_\rho)) \\
&\quad - \frac{1}{2} (\partial_\mu \partial_\rho (h_{\sigma\nu} - \partial_\sigma \xi_\nu - \partial_\nu \xi_\sigma) + \partial_\nu \partial_\rho (h_{\sigma\mu} - \partial_\sigma \xi_\mu - \partial_\mu \xi_\sigma)) \\
&= R_{\rho\sigma\mu\nu} + \frac{1}{2} (-\partial_\mu \partial_\sigma \partial_\rho \xi_\nu - \partial_\mu \partial_\sigma \partial_\nu \xi_\rho + \partial_\nu \partial_\sigma \partial_\rho \xi_\mu + \partial_\nu \partial_\sigma \partial_\mu \xi_\rho) \\
&\quad + \frac{1}{2} (\partial_\mu \partial_\rho \partial_\sigma \xi_\nu + \partial_\mu \partial_\rho \partial_\nu \xi_\sigma - \partial_\nu \partial_\rho \partial_\sigma \xi_\mu - \partial_\nu \partial_\rho \partial_\mu \xi_\sigma) \\
&= R_{\rho\sigma\mu\nu}
\end{aligned} \tag{16}$$

This is akin to gauge transformation of electromagnetic potential $A'_\mu = A_\mu + \partial_\mu \alpha$ that leaves the EM field strength (curvature) tensor invariant

$$F'_{\mu\nu} = \partial_\mu A'_\nu - \partial_\nu A'_\mu = \partial_\mu (A_\nu + \partial_\nu \alpha) - \partial_\nu (A_\mu + \partial_\mu \alpha) = \partial_\mu A_\nu - \partial_\nu A_\mu = F_{\mu\nu} \tag{17}$$

1.3 Metric Perturbation Example

Exercise: Consider the metric perturbation

$$h_{\mu\nu} = \begin{pmatrix} c & 0 & 0 & 0 \\ 0 & -c & 0 & 0 \\ 0 & 0 & a & 0 \\ 0 & 0 & 0 & -a \end{pmatrix} \sin(k(x-t)) \tag{18}$$

for arbitrary constants a and c .

1. Show that $h_{\mu\nu}$ solves the linearized Einstein equations in vacuum (14).
2. Find the functions $\xi_\mu(x)$ that transform it into a diagonal form where $c = 0$. What is the new value of a ?

1.3.1 Show that $h_{\mu\nu}$ is a solution

Ricci tensor (5) has two kinds of terms, $\square h_{\mu\nu}$ and V_μ (6).

$$\begin{aligned}
\square h_{\mu\nu} &= (-\partial_t^2 + \partial_x^2 + \partial_y^2 + \partial_z^2) \sin(k(x-t)) \\
&= -(-k^2 \sin(k(x-t))) + (-k^2 \sin(k(x-t))) = 0
\end{aligned} \tag{19}$$

The trace $h_{\mu\nu}$ is

$$\begin{aligned} h &= \eta^{\rho\sigma} h_{\rho\sigma} = -h_{tt} + h_{xx} + h_{yy} + h_{zz} \\ &= [-(c) + (-c) + a + (-a)] \sin(k(x-t)) = -2c \sin(k(x-t)) \end{aligned} \quad (20)$$

Notice it is not zero. We need to calculate

$$V_\mu = \partial_\nu h^\nu_\mu - \frac{1}{2} \partial_\mu h = \eta^{\nu\rho} \partial_\nu h_{\rho\mu} - \frac{1}{2} \partial_\mu h \quad (21)$$

Since $\eta^{\nu\rho}$ and $h_{\rho\mu}$ are diagonal, we find

$$V_t = \eta^{tt} \partial_t h_{tt} - \frac{1}{2} \partial_t h = \partial_t \left(-h_{tt} - \frac{1}{2} h \right) = \partial_t \left[\left(-c - \frac{1}{2} (-2c) \right) \sin(k(x-t)) \right] = 0 \quad (22)$$

$$V_x = \eta^{xx} \partial_x h_{xx} - \frac{1}{2} \partial_x h = \partial_x \left[h_{xx} - \frac{1}{2} h \right] = \partial_x \left[\left((-c) - \frac{1}{2} (-2c) \right) \sin(k(x-t)) \right] = 0 \quad (23)$$

$$V_y = 0 \quad V_z = 0 \quad (24)$$

We found that indeed

$$(R_{\mu\nu})^{(1)} = \frac{1}{2} (\partial_\mu V_\nu + \partial_\nu V_\mu - \square h_{\mu\nu}) = 0 \quad (25)$$

1.3.2 Find the functions $\xi_\mu(x)$

We demand that the gauge transformation (11) would yield $h'_{tt} = 0$

$$0 \stackrel{!}{=} h'_{tt} = h_{tt} - 2\partial_t \xi_t \quad (26)$$

\Rightarrow

$$\partial_t \xi_t = \frac{1}{2} h_{tt} = \frac{c}{2} \sin(k(x-t)) \quad (27)$$

integrate

$$\xi_t = \frac{c}{2k} \cos(k(x-t)) \quad (28)$$

We also demand that the gauge transformation (11) would yield $h'_{xx} = 0$

$$0 \stackrel{!}{=} h'_{xx} = h_{xx} - 2\partial_x \xi_x \quad (29)$$

⇒

$$\partial_x \xi_x = \frac{1}{2} h_{xx} = \frac{-c}{2} \sin(k(x-t)) \quad (30)$$

integrate

$$\xi_x = \frac{c}{2k} \cos(k(x-t)) \quad (31)$$

Check that it does not create non diagonal element h'_{tx}

$$h'_{tx} = h_{tx} - \partial_t \xi_x - \partial_x \xi_t = -(\partial_t + \partial_x) \frac{c}{2k} \cos(k(x-t)) = 0 \quad (32)$$

Take

$$\xi_y = \xi_z = 0 \quad (33)$$

and since ξ_t and ξ_x do not depend on y and z , $\partial_y \xi_\mu = \partial_z \xi_\mu = \partial_\mu \xi_y = \partial_\mu \xi_z = 0$. Therefore no components of $h_{\mu\nu}$ change except for h_{tt} and h_{xx} . "a" does not change.

2 Production Of Gravitational Waves

2.1 Linearized Einstein Equation With Sources

Define the *trace-reversed* perturbation

$$\bar{h}_{\mu\nu} := h_{\mu\nu} - \frac{1}{2} \eta_{\mu\nu} h \quad (34)$$

Exercise: Justify the name "trace-reversed" for $\bar{h}_{\mu\nu}$.

$$\eta^{\mu\nu} \bar{h}_{\mu\nu} = \eta^{\mu\nu} \left(h_{\mu\nu} - \frac{1}{2} \eta_{\mu\nu} h \right) = h - \frac{1}{2} 4h = -h \quad (35)$$

$\bar{h}_{\mu\nu}$ has the negative trace of $h_{\mu\nu}$ (in four dimensions). In the transverse-traceless gauge they coincide.

Exercise: Write the Lorenz gauge condition (12) for $\bar{h}_{\mu\nu}$.

$$\partial_\nu h^{\nu\mu} - \frac{1}{2} \partial^\mu h = \partial_\nu \left(h^{\nu\mu} - \frac{1}{2} \eta^{\nu\mu} h \right) = \partial_\nu \bar{h}^{\mu\nu} = 0 \quad (36)$$

Exercise: Write the linearized Einstein equations with sources in Lorenz gauge.

In Lorenz gauge the Ricci tensor (5) is

$$(R_{\mu\nu})^{(1)} = -\frac{1}{2} \square h_{\mu\nu} \quad (37)$$

and the Ricci scalar (9) is

$$R^{(1)} = \partial_\mu \partial_\nu h^{\mu\nu} - \partial_\mu \partial^\mu h = \partial_\mu \left(\frac{1}{2} \partial^\mu h \right) - \partial_\mu \partial^\mu h = -\frac{1}{2} \square h \quad (38)$$

so the Einstein tensor is

$$\begin{aligned} (G_{\mu\nu})^{(1)} &= (R_{\mu\nu})^{(1)} - \frac{1}{2} \eta_{\mu\nu} R^{(1)} \\ &= -\frac{1}{2} \left(\square h_{\mu\nu} - \frac{1}{2} \eta_{\mu\nu} \square h \right) = -\frac{1}{2} \square \left(h_{\mu\nu} - \frac{1}{2} \eta_{\mu\nu} h \right) = -\frac{1}{2} \square \bar{h}_{\mu\nu} \end{aligned} \quad (39)$$

The linearized Einstein equations with sources in Lorenz gauge is

$$\square \bar{h}_{\mu\nu} = -16\pi T_{\mu\nu} \quad (40)$$

Linearized Einstein equation with source in Lorenz gauge

(40) are wave equations with sources for each $\bar{h}_{\mu\nu}$. The vacuum equations have Ricci tensor on the l.f.h, and are wave equations for $h_{\mu\nu}$ in Lorenz gauge (15). The equations with sources in Lorenz gauge turn out to be wave equations for $\bar{h}_{\mu\nu}$ since in the l.h.s there is the Einstein tensor, which is the “trace-reversed Ricci tensor”.

2.2 The General Solution

Write down the general solution of (40).

This step is exactly like in electromagnetism, so we quote in short.

The Green function $G(x^\sigma - y^\sigma)$ of the \square operator is the solution to a point (delta function) source

$$\square_x G(x^\sigma - y^\sigma) = \delta^{(4)}(x^\sigma - y^\sigma) \quad (41)$$

where y^σ is the position of the source and x^σ is where we evaluate the field (in spacetime). The *retarded* solution is

$$G(x^\sigma - y^\sigma) = -\frac{1}{4\pi |\mathbf{x} - \mathbf{y}|} \delta[|\mathbf{x} - \mathbf{y}| - (x^0 - y^0)] \theta(x^0 - y^0) \quad (42)$$

where \mathbf{x}, \mathbf{y} are three-vectors, $x^0 \equiv t$, θ is Heaviside function. The general

solution is an integral over the source with the Green function

$$\bar{h}_{\mu\nu}(x^\sigma) = -16\pi \int G(x^\sigma - y^\sigma) T_{\mu\nu}(y^\sigma) d^4y \quad (43)$$

(apply \square_x to both sides of (43) and with the Green function definition (41) recover (40)).

We perform the integral in (43) over dt , by the delta function in (42), and get

$$\bar{h}_{\mu\nu}(x^\sigma) = 4 \int \frac{1}{|\mathbf{x} - \mathbf{y}|} T_{\mu\nu}(t - |\mathbf{x} - \mathbf{y}|, \mathbf{y}) d^3y \quad (44)$$

where we substituted $y^0 = x^0 - |\mathbf{x} - \mathbf{y}| = t - |\mathbf{x} - \mathbf{y}|$. The retarded time is

$$t_r = t - |\mathbf{x} - \mathbf{y}| \quad (45)$$

since $|\mathbf{x} - \mathbf{y}|$ times $c = 1$ is the time it takes the signal from the source at \mathbf{y} to reach \mathbf{x} . The general solution can be written as

$$\bar{h}_{\mu\nu}(t, \mathbf{x}) = 4 \int \frac{1}{|\mathbf{x} - \mathbf{y}|} T_{\mu\nu}(t_r, \mathbf{y}) d^3y \quad (46)$$

General solution
to wave equations
with source

2.3 An Approximate Solution

Use a time-Fourier transform and approximate (46) for a source which is isolated, far away, and slowly moving (non-relativistic).

First we make a time - Fourier transform to (46)

$$\begin{aligned} \tilde{\bar{h}}_{\mu\nu}(\omega, \mathbf{x}) &= \frac{1}{\sqrt{2\pi}} \int dt e^{i\omega t} \bar{h}_{\mu\nu}(t, \mathbf{x}) \\ &= \frac{4}{\sqrt{2\pi}} \int dt d^3y e^{i\omega t} \frac{1}{|\mathbf{x} - \mathbf{y}|} T_{\mu\nu}(t - |\mathbf{x} - \mathbf{y}|, \mathbf{y}) \\ &= \frac{4}{\sqrt{2\pi}} \int dt_r d^3y e^{i\omega t_r} \frac{e^{i\omega|\mathbf{x}-\mathbf{y}|}}{|\mathbf{x} - \mathbf{y}|} T_{\mu\nu}(t_r, \mathbf{y}) \\ &= 4 \int d^3y \frac{e^{i\omega|\mathbf{x}-\mathbf{y}|}}{|\mathbf{x} - \mathbf{y}|} \tilde{T}_{\mu\nu}(\omega, \mathbf{y}) \end{aligned} \quad (47)$$

Now we make the approximations, which mean that we consider a source centered at spatial distance r with, which is confined to a radius $\delta r \ll r$. Slowly moving source would emit low frequencies. $\delta r \ll \omega^{-1} \sim \lambda$ (long wavelength ap-

proximation). So, $|\mathbf{x} - \mathbf{y}| \sim r$ and $\omega |\mathbf{x} - \mathbf{y}| \sim \omega r$, and we can replace $\frac{e^{i\omega|\mathbf{x}-\mathbf{y}|}}{|\mathbf{x}-\mathbf{y}|}$ by $\frac{e^{i\omega r}}{r}$ and take it out of the integral.

$$\tilde{h}_{\mu\nu}(\omega, \mathbf{x}) = 4 \frac{e^{i\omega r}}{r} \int d^3y \tilde{T}_{\mu\nu}(\omega, \mathbf{y}) \quad (48)$$

Transform back to

$$\bar{h}_{\mu\nu}(t, \mathbf{x}) = \frac{4}{r} \int d^3y T_{\mu\nu}(t_r, \mathbf{y}) \quad (49)$$

Weak, isolated,
far and slow
source solution

where now

$$t_r = t - r \quad (50)$$

2.4 Tensor Virial Theorem

Exercise 1: Show that in flat spacetime $\partial_i \partial_j T^{ij} = \partial_t^2 T^{tt}$.

Energy-momentum conservation in flat spacetime is

$$\partial_\mu T^{\mu\nu} = 0 \quad (51)$$

The t and i components explicitly are

$$\partial_t T^{tt} + \partial_i T^{it} = 0 \quad (52)$$

$$\partial_t T^{ti} + \partial_j T^{ji} = 0 \quad (53)$$

Using (53), commute partial derivatives, and then (52), yield

$$\partial_i \partial_j T^{ij} = -\partial_i \partial_t T^{ti} = -\partial_t \partial_i T^{ti} = \partial_t \partial_t T^{tt} \quad (54)$$

Exercise 2: Show that

$$\int d^3y T^{ij} = \frac{1}{2} \frac{d^2}{dt^2} \int d^3y T^{tt} y^i y^j \quad (55)$$

We multiply (54) by $y^i y^j$ and integrate over $\int d^3y$. Use integration by parts,

and that the source is isolated so the surface terms vanish, the l.h.s is

$$\begin{aligned}
 \int dy^3 y^i y^j \partial_k \partial_l T^{kl} &= - \int dy^3 \partial_k (y^i y^j) \partial_l T^{kl} = - \int dy^3 (\delta_k^i y^j + y^i \delta_k^j) \partial_l T^{kl} \\
 &= - \int dy^3 (y^j \partial_l T^{il} + y^i \partial_l T^{jl}) = \int dy^3 ((\partial_l y^j) T^{il} + (\partial_l y^i) T^{jl}) \\
 &= \int dy^3 (\delta_l^j T^{il} + \delta_l^i T^{jl}) = \int dy^3 (T^{ij} + T^{ji}) = 2 \int dy^3 T^{ij}
 \end{aligned} \tag{56}$$

Together with the r.h.s, (54) becomes

$$\int dy^3 T^{ij} = \frac{1}{2} \frac{d^2}{dt^2} \int dy^3 T^{tt} y^i y^j \tag{57}$$

Tensor virial theorem

2.5 The Quadrupole Formula

Define the quadrupole moment tensor of the energy density of the source

$$I_{ij}(t) = \int dy^3 T^{tt} y_i y_j \tag{58}$$

Quadrupole moment tensor

With (58), (57) becomes

$$\int dy^3 T^{ij} = \frac{1}{2} \frac{d^2 I_{ij}}{dt^2} \tag{59}$$

and the approximate solution to the spatial components of the metric perturbation (49) becomes

$$\bar{h}_{ij}(t, \mathbf{x}) = \frac{2}{r} \frac{d^2 I_{ij}}{dt^2}(t_r) \tag{60}$$

The quadrupole formula

What about the remaining components of the metric perturbation, $h^{t\mu}$? They can be deduced from h^{ij} and the Lorenz gauge condition.

3 Gravitational Radiation From Binary Stars

3.1 Metric Perturbation

Exercise: Consider a simple binary star (two stars orbiting each other). It consists of two stars of mass M in a circular orbit in the $x^1 - x^2$ plane ,at

distance R from their common center of mass. Calculate $\bar{h}_{ij}(t, \mathbf{x})$ in the weak, isolated, far, slow approximation, and for Newtonian orbit of the stars.

The Newton second law for a mass M in a circular orbit of radius R , under the gravitational force of another mass M at distance $2R$, is

$$\frac{M^2}{(2R)^2} = M\Omega^2 R \quad (61)$$

where Ω is the **orbital** frequency (velocity). It reads

$$\Omega = \left(\frac{M}{4R^3} \right)^{\frac{1}{2}} \quad (62)$$

The positions of star a is

$$x_a^1 = R \cos(\Omega t) \quad (63)$$

$$x_a^2 = R \sin(\Omega t) \quad (64)$$

$$x_a^3 = 0 \quad (65)$$

The position of star b is the negative coordinates

$$x_b^1 = -R \cos(\Omega t) \quad (66)$$

$$x_b^2 = -R \sin(\Omega t) \quad (67)$$

$$x_b^3 = 0 \quad (68)$$

The quadrupole moment for this discrete system can be written as

$$\begin{aligned} I_{ij} &= \sum_c m_c x_c^i x_c^j = M x_a^i x_a^j + M x_b^i x_b^j = M x_a^i x_a^j + M (-x_a^i) (-x_a^j) \\ &= 2M x_a^i x_a^j \end{aligned} \quad (69)$$

The components of the quadrupole moment tensor are

$$I_{11} = 2MR^2 \cos^2(\Omega t) = MR^2 (1 + \cos(2\Omega t)) \quad (70)$$

$$I_{22} = 2MR^2 \sin^2(\Omega t) = MR^2 (1 - \cos(2\Omega t)) \quad (71)$$

$$I_{12} = 2MR^2 \sin(\Omega t) \cos(\Omega t) = MR^2 \sin(2\Omega t) \quad (72)$$

$$I_{13} = I_{23} = I_{33} = 0 \quad (73)$$

Differentiate (70),(71)(72) twice, and plug into (60)

$$\bar{h}_{ij}(t, \mathbf{x}) = \frac{8}{r} M \Omega^2 R^2 \begin{pmatrix} -\cos(2\Omega t_r) & -\sin(2\Omega t_r) & 0 \\ -\sin(2\Omega t_r) & \cos(2\Omega t_r) & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad (74)$$

The radiation frequency $\omega = 2\Omega$ is twice the orbital frequency, since the total binary star reach the same configuration after half of the period of its stars.

3.2 Energy Loss Due To Gravitational Radiation

Define a *reduced/traceless quadrupole moment* as

$$J_{ij} = I_{ij} - \frac{1}{3} I \delta_{ij} \quad (75)$$

where I is the trace of I_{ij}

$$I \equiv \delta^{kl} I_{kl} \quad (76)$$

Explicitly,

$$J_{ij} = \int d^3y T^{tt} \left(y_i y_j - \frac{1}{3} y^2 \delta_{ij} \right) \quad (77)$$

This is the actual quadrupole moment that appears in a multipole expansion of Newtonian potential.

The power radiated in gravitational waves is

$$P = -\frac{1}{5} \left\langle \frac{d^3 J_{ij}}{dt^3} \frac{d^3 J^{ij}}{dt^3} \right\rangle \quad (78)$$

**Traceless
quadrupole
formula for
radiated power**

where the quadrupole moment is evaluated at the retarded time $t - r$, and the $\langle \dots \rangle$ denotes the time average over a period. We will not derive this formula, yet here are few remarks on some logic in this formula: Energy is proportional to amplitude squared of a wave; Power is energy/time and is dimensionless in geometrical units, and so is $\frac{d^3 J_{ij}}{dt^3}$; A spherically symmetric source emits no radiation, and likewise it has only monopole in the multipole expansion, so $J_{ij} = 0$ (unlike I_{ij}).

Exercise: Calculate the power emitted in radiation by a binary star.

First we compute J_{ij} for (70)-(73). The trace I is

$$I = I_{11} + I_{22} + I_{33} = 2MR^2 \quad (79)$$

Plug in (75)

$$J_{ij} = \frac{MR^2}{3} \begin{pmatrix} 1 + 3\cos(2\Omega t) & 3\sin(2\Omega t) & 0 \\ 3\sin(2\Omega t) & 1 - 3\cos(2\Omega t) & 0 \\ 0 & 0 & -2 \end{pmatrix} \quad (80)$$

$$\frac{d^3 J_{ij}}{dt^3} = 8MR^2\Omega^3 \begin{pmatrix} \sin(2\Omega t) & -\cos(2\Omega t) & 0 \\ -\cos(2\Omega t) & -\sin(2\Omega t) & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad (81)$$

by matrix multiplication and trace

$$\frac{d^3 J_{ij}}{dt^3} \frac{d^3 J^{ij}}{dt^3} = (8MR^2\Omega^3)^2 \text{Tr} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} = 128M^2R^4\Omega^6 \quad (82)$$

No time average is needed in this case, plug in (78)

$$P = -\frac{128}{5}M^2R^4\Omega^6 \quad (83)$$

and only in terms of M, R , plug Ω (62)

$$P = -\frac{2}{5} \left(\frac{M}{R} \right)^5 \quad (84)$$

Power emitted in
radiation form a
binary star