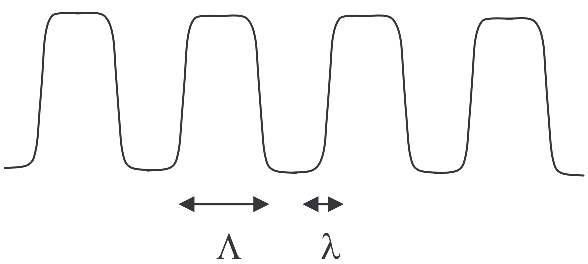
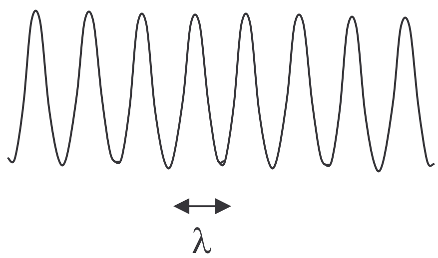
Spatially Extended Bistable Systems

A. Domain patterns

Up to now we considered patterns that emerge from instabilities of uniform zero states, such as the stationary finite-k instability in the SH model that leads to stationary stripe patterns, or the uniform Hopf bifurcation that leads to long-wavelength traveling waves. Patterns of this kind are characterized by a single length scale – the pattern wavelength. Another type of patterns appears in spatially extended *bistable systems*, that is, systems having two stable uniform states. Such systems support patterns consisting of alternate domains of the two stable states as illustrated in the figure below (on the left). We will refer to patterns of this type as to *domain patterns*.





Unlike patterns emerging from instabilities of uniform states (figure on the right), domain patterns are characterized by two length scales: the domain size Λ and the usually much shorter width λ of the interface between domains of different states.

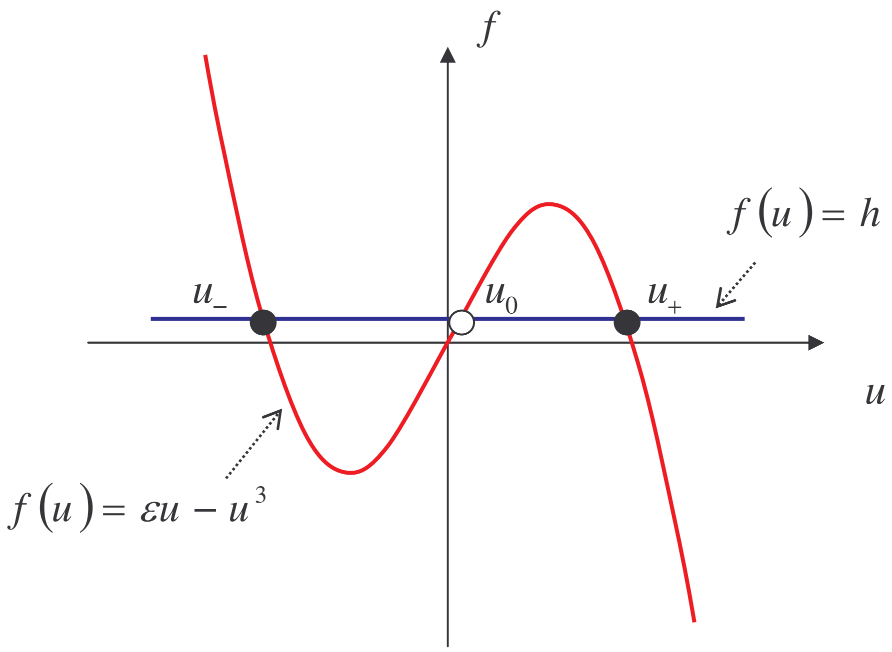
The properties of domain patterns are strongly affected by the structure and dynamics of the interfaces between the two different states. We will refer to these interfaces as to *fronts*.

B. Gradient bistable system

The simplest model for a spatially extended bistable system is the time-dependent Ginzburg Landau equation (TDGL) for a real field $u(x,t)$,

$$\frac{\partial u}{\partial t} = \varepsilon u - u^3 - h + D\frac{\partial^2 u}{\partial x^2}, \qquad (1)$$

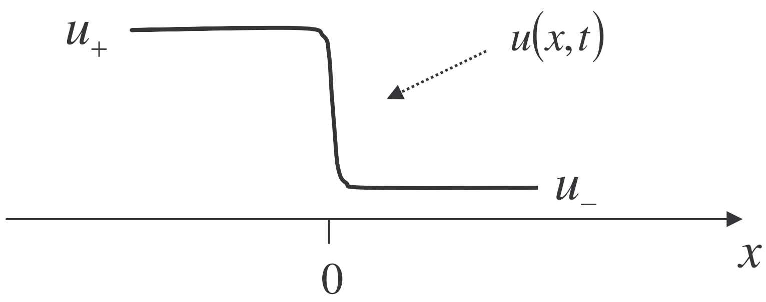
where h is a real constant. The stationary uniform states of this system can be studied graphically by plotting the function $f(u) = \varepsilon u - u^3$ and intersecting it with the constant function $f(u) = h$. The intersection points give the stationary uniform states of (1). For negative ε values the function $f(u) = \varepsilon u - u^3$ is monotonically decreasing and there can be only one intersection point with $f(u) = h$, or only one stationary uniform state. A bistable system can therefore appear only for $\varepsilon > 0$ as illustrated in the figure below.



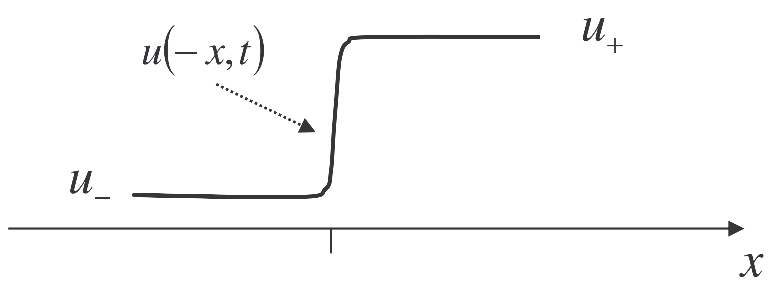
Of the three intersection points, u_-, u_0, u_+, only the outer two (solid circles) represent stable stationary uniform states as can be verified by a linear stability analysis. Since we are interested in positive values of ε only, we can rescale the time coordinate, $t \to \varepsilon t$, and eliminate ε from Eq. (1). Similarly we can rescale the space coordinate $x \to x/\sqrt{D}$ and eliminate D. This leads to

$$\frac{\partial u}{\partial t} = u - u^3 - h + \frac{\partial^2 u}{\partial x^2}, \qquad (2)$$

Imagine now preparing the left half of the system ($-\infty < x < 0$) in the u_+ state and the right half ($0 < x < \infty$) in the u_- state. Such an initial condition will lead to a front solution biasymptotic to u_+ and u_-. The diffusion term will smooth out the initial discontinuity at $x = 0$. A schematic graph of this front solution is shown in the figure below.



Equation (1) is invariant under reflections $x \to -x$ because it does not have any explicit dependence on x and it contains only even derivatives with respect to x. As a result, if $u(x,t)$ is a solution also $u(-x,t)$ is a solution:



Using this pair of symmetric front solutions we can "construct" patterns that look like

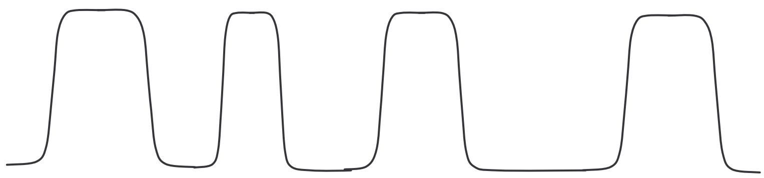


Fig. 1

Will such an initial condition for Eq. (1) lead to a stable asymptotic pattern?

To answer this question we will show that Eq. (1) is *gradient*. By this we mean that there exists a functional $L[u(x,t)]$ in terms of which Eq. (2) can be written as

$$\frac{\partial u}{\partial t} = -\frac{\delta L}{\delta u}, \qquad (3)$$

where δ denotes the variational derivative. We will shortly prove that this functional is given by

$$L[u] = \int_{-\infty}^{\infty} \left[E(u) + \frac{1}{2}\left(\frac{\partial u}{\partial x'}\right)^2 \right] dx'$$

$$E = -\frac{1}{2}u^2 + \frac{1}{4}u^4 + hu \qquad (4)$$

but we first want to discuss the significance of being able to write Eq. (2) in the form of (3). Let us take the total time derivative of $L[u(x,t)]$. Since

L operates on a continuum of functions $u(t)$ indexed by x we should apply the variational form of the chain rule and write:

$$\frac{dL}{dt} = \int_{-\infty}^{\infty} \frac{\delta L}{\delta u} \frac{\partial u}{\partial t} dx' = -\int_{-\infty}^{\infty} \left(\frac{\partial u}{\partial t} \right)^2 dx' \leq 0. \qquad (5)$$

We thus find that L cannot increase in time! The significance of this result is that if L has a minimum, the time evolution of u will be such as to minimize L until it reaches that minimum. Once L attained its minimum no further time evolution can take place and the system's state becomes stationary. The functional L is called a *Lyapunov function* or sometimes an *energy function*.

Let us show now that Eq. (2) can be written in the form (3). To this end consider a variation of L obtained by varying the function u subject to the constraint $\delta u(\pm\infty, t) = 0$:

$$\delta L = \int_{-\infty}^{\infty} \left\{ \delta E + \delta \left[\frac{1}{2} \left(\frac{\partial u}{\partial x'} \right)^2 \right] \right\} dx' = \int_{-\infty}^{\infty} \left[\frac{\delta E}{\delta u} \delta u + \frac{\partial u}{\partial x'} \delta \left(\frac{\partial u}{\partial x'} \right) \right] dx' =$$

$$\int_{-\infty}^{\infty} \left[\frac{\delta E}{\delta u} \delta u + \frac{\partial u}{\partial x'} \frac{\partial}{\partial x'} (\delta u) \right] dx' = \int_{-\infty}^{\infty} \left[\frac{\delta E}{\delta u} \delta u - \frac{\partial^2 u}{\partial x'^2} \delta u \right] dx'$$

Integration by parts

We thus obtained

$$\delta L = \int_{-\infty}^{\infty} \left(\frac{\delta E}{\delta u} - \frac{\partial^2 u}{\partial x'^2} \right) \delta u(x', t) dx'$$

from which we deduce

$$\frac{\delta L}{\delta u(x,t)} = \frac{\delta E}{\delta u} - \frac{\partial^2 u}{\partial x^2} \qquad (6)$$

Using (4) in (6) and we find that Eq. (3) is equivalent to Eq. (2).

Let us find now the minima of L. From (4), the minima of E are solutions of $u - u^3 - h = 0$ or, in other words, the stationary uniform solutions of Eq. (2). These are also the minima of L because the uniform solutions minimize the second term of L, $(1/2)(\partial u/\partial x)^2$. We can answer now the question we asked: what asymptotic state the initial condition shown in Fig. 1 will evolve to? The answer is the uniform state that has lower energy E. If $h > 0$, $E(u_-) < E(u_+)$ and the system will evolve to u_-. If $h < 0$ the system will evolve to u_+.

How does the evolution from the initial condition of Fig. 1 to one of the uniform states u_+ or u_- takes place? To answer this question we first analyze a single front solution. We now understand that the front will propagate so as to minimize the energy function. To find the front velocity c, we consider solutions of the form $u(x,t) = u(\chi)$, $\chi = x - ct$, describing fronts that propagate at constant velocities (note that $c > 0$ implies propagation in the positive χ direction). We further assume that

$$\lim_{\chi \to \pm\infty} u = u_{\mp} \,. \tag{7}$$

Inserting $u = u(\chi)$ into (1) we obtain

$$u'' + cu' + u - u^3 - h = 0\,, \tag{8}$$

where the prime denotes derivation with respect to χ. Multiplying Eq. (8) by u' and integrating over the entire line $-\infty < \chi < \infty$ we find

$$c = \frac{E(u_-) - E(u_+)}{\int_{-\infty}^{\infty} u'^2 d\chi}. \tag{9}$$

If instead of (7) we would have assumed

$$\lim_{\chi \to \pm\infty} u = u_{\pm} \tag{10}$$

we would have found

$$c = \frac{E(u_+) - E(u_-)}{\int_{-\infty}^{\infty} u'^2 d\chi}. \tag{11}$$

The explicit form of a front solution satisfying (7) is

$$u(\chi) = \frac{u_- e^{\alpha\chi} + u_+}{1 + e^{\alpha\chi}}, \tag{12}$$

where

$$\alpha = \frac{1}{\sqrt{2}}(u_+ - u_-) \qquad\qquad c = \frac{1}{\sqrt{2}}(u_+ - 2u_0 + u_-)$$

and we recall that u_-, u_0, u_+ are the 3 solutions of $u - u^3 - h = 0$.

Using (9) and (11) we can now anticipate the dynamics of the initial condition of Fig. 1 towards one of the uniform stationary states $u_{\pm}$. Assuming for instance that $h > 0$ the fronts that make the initial pattern in Fig.1 will follow the following directions

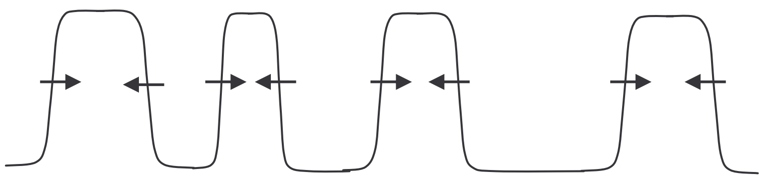


Fig. 2

And the pattern will converge to the uniform state u_-. In other words, the u_+ or up-state domains will shrink in size and eventually disappear leaving the whole system in the down-state u_-. Conversely, if $h < 0$ the down-state domains will shrink in size leaving the system in the up-state u_+. What do you expect to happen in the limiting case $h = 0$? Can you deduce if nearby fronts attract or repel one another?