Brownian motion of a particle in a potential

The Langevin equation describing the dynamics of a Brownian particle moving in a potential $U(x)$ is

$$\frac{dx}{dt} = \frac{p}{m}; \quad \frac{dp}{dt} = -U'(x) - \varsigma \frac{p}{m} + F_p(t) \tag{1}$$

and the fluctuation dissipation theorem is

$$\langle F_p(t) F_p(t') \rangle = 2\varsigma kT \delta(t - t'). \tag{2}$$

The quantities that go into the general Fokker-Planck equation are

$$\begin{aligned} \mathbf{a} &= \begin{pmatrix} x \\ p \end{pmatrix}; \quad \mathbf{v}(\mathbf{a}) = \begin{pmatrix} p/m \\ -U'(x) - \varsigma \frac{p}{m} \end{pmatrix}; \\ F(t) &= \begin{pmatrix} 0 \\ F_p(t) \end{pmatrix}; \quad \mathbf{B} = \begin{pmatrix} 0 & 0 \\ 0 & \varsigma kT \end{pmatrix}; \end{aligned}$$

The Fokker-Planck equation may be written as

$$\begin{aligned} \frac{\partial f(x,p,t)}{\partial t} &= -\frac{\partial}{\partial x}\left(\frac{p}{m} f(x,p,t)\right) - \frac{\partial}{\partial p}\left(\left(-U'(x) - \varsigma \frac{p}{m}\right) f(x,p,t)\right) \\ &\quad + \varsigma kT \frac{\partial^2}{\partial p^2} f(x,p,t). \end{aligned} \tag{3}$$

$$\begin{aligned} \frac{\partial f(x,p,t)}{\partial t} &= \frac{\varsigma}{m} f(x,p,t) - \frac{p}{m}\frac{\partial}{\partial x} f(x,p,t) + \left(U'(x) + \frac{\varsigma p}{m}\right)\frac{\partial}{\partial p} f(x,p,t) \\ &\quad + \varsigma kT \frac{\partial^2}{\partial p^2} f(x,p,t). \end{aligned} \tag{4}$$

It is possible to show that the equilibrium solution is

$$f_{eq}(x,p) = \frac{1}{Q} e^{-H(x,p)/kT} \text{ where } Q = \iint dx dp e^{-H(x,p)/kT},$$

and

$$H = \frac{p^2}{2m} + U(x).$$

Note that Q, is the partition function and $f_{eq}(x,p)$ is the Boltzmann distribution.

If we consider the over-damped dynamics, the one for which the acceleration term is negligible, we obtain the following Langevin equation

$$\frac{dx}{dt} = -\frac{1}{\varsigma} U'(x) + \frac{1}{\varsigma} F_p(t).$$

In this case the quantities to be used in the general expression for the Fokker-Planck equation are

$$\begin{aligned}\mathbf{a} &= x; \quad \mathbf{v}(\mathbf{a}) = -\frac{1}{\varsigma}U'(x); \\ F(t) &= \frac{1}{\varsigma}F_p(t); \quad \mathbf{B} = \frac{kT}{\varsigma} = D;\end{aligned}$$

Substituting these quantities in equation (**??**) we obtain the Smoluchowski equation

$$\frac{\partial f(x,t)}{\partial t} = \frac{1}{\varsigma}\frac{\partial}{\partial x}\left(U'(x) f(x,t)\right) + \frac{kT}{\varsigma}\frac{\partial^2}{\partial x^2} f(x,t) \quad (5)$$

$$= D\frac{\partial}{\partial x}\left[e^{-U(x)/kT}\frac{\partial}{\partial x}\left(e^{U(x)/kT} f(x,t)\right)\right], \quad (6)$$

where the diffusion coefficient is

$$D = \frac{kT}{\varsigma}.$$

General properties of Fokker-Planck equations

- Parabolic differential equations. A general PDE of the form

  $$Au_{xx} + Bu_{xy} + Cu_{yy} + Du_x + Eu_y + F = 0$$

  is parabolic if it satisfies the condition

  $$B^2 - 4AC = 0.$$

- There is no guarantee of a steady state solution.
- Generally non self adjoint and therefore little is known about their mathematical properties.
- It is likely that they can have eigenfunctions and eigenvalues. However, these can be found only for special cases. There are no general theorems about the existence or completeness of eigenfunction expansions. An exception to this rule is the Smoluchowski equation which can be made self-adjoint by a trick. The substitution

  $$f = \sqrt{f_{eq}}g$$

  leads to a Schrödinger like equation for g

  $$\begin{aligned}-\frac{\partial g}{\partial t} &= D\left(-\frac{\partial^2}{\partial x^2} + U_{eff}(x)\right)g \\ U_{eff}(x) &= \left(\frac{1}{2kT}\frac{\partial U}{\partial x}\right)^2 - \frac{1}{2kT}\frac{\partial^2 U}{\partial x^2}. \\ D &= \frac{kT}{\varsigma}\end{aligned}$$

The properties of this equation are well known. It has real eigenvalues and eigenfunctions which form a complete set. Solutions for a specific potential may be difficult to find, however, there is no conceptual difficulty. For the more general Fokker-Planck the same trick does not lead to a self adjoint equation.

Averages

Sometimes we are only interested in certain averages. These can be found by two distinct but equivalent methods. We can follow the evolution of some initial state using the Fokker-Planck equation

$$\frac{\partial f(\mathbf{a},t)}{\partial t}=\mathcal{D}f(\mathbf{a},t)\,,$$

with the operator $\mathcal{D}$ defined by

$$\mathcal{D}\equiv-\frac{\partial}{\partial\mathbf{a}}\cdot\mathbf{v}(\mathbf{a})+\frac{\partial}{\partial\mathbf{a}}\cdot\left(\mathbf{B}\cdot\frac{\partial}{\partial\mathbf{a}}\right).$$

The formal solution may be written as

$$f(\mathbf{a},t)=e^{t\mathcal{D}}f(\mathbf{a},0)\,.$$

The average of any dynamical quantity $\phi(\mathbf{a})$ is given by

$$\langle\phi(\mathbf{a})\rangle=\int d\mathbf{a}\phi(\mathbf{a})\,f(\mathbf{a},t)=\int d\mathbf{a}\phi(\mathbf{a})\,e^{t\mathcal{D}}f(\mathbf{a},0)\,.$$

The second method uses the adjoint operator of $\mathcal{D}$ defined by

$$\begin{aligned}\int d\mathbf{a}\phi(\mathbf{a})\,\mathcal{D}\psi(\mathbf{a}) &= \int d\mathbf{a}\psi(\mathbf{a})\,\mathcal{D}^{+}\phi(\mathbf{a})\\ \mathcal{D}^{+} &\equiv \mathbf{v}(\mathbf{a})\cdot\frac{\partial}{\partial\mathbf{a}}+\frac{\partial}{\partial\mathbf{a}}\cdot\left(\mathbf{B}\cdot\frac{\partial}{\partial\mathbf{a}}\right).\end{aligned}$$

Now the average can be obtained by reversing the operator in the exponent

$$\begin{aligned}\langle\phi(\mathbf{a})\rangle &= \int d\mathbf{a}f(\mathbf{a},0)\,e^{t\mathcal{D}^{+}}\phi(\mathbf{a})=\int d\mathbf{a}f(\mathbf{a},0)\,\phi(\mathbf{a},t)\,;\\ \phi(\mathbf{a},t) &\equiv e^{t\mathcal{D}^{+}}\phi(\mathbf{a})\,.\end{aligned}$$

The equation of motion for $\phi(\mathbf{a},t)$ is

$$\frac{\partial\phi(\mathbf{a},t)}{\partial t}=\mathcal{D}^{+}\phi(\mathbf{a},t)\,.$$

For instance, if we consider the Smoluchowski equation for which we found

$$\begin{aligned}\mathbf{a} &= x;\quad \mathbf{v}(\mathbf{a})=-\frac{1}{\varsigma}U'(x)\,;\qquad \mathbf{B}=\frac{kT}{\varsigma}=D;\\ \mathcal{D}^{+} &= -\frac{1}{\varsigma}U'(x)\,\frac{\partial}{\partial x}+D\frac{\partial^2}{\partial x^2};\end{aligned}$$

the equation for the average of any function of x is

$$\frac{\partial \phi(x,t)}{\partial t} = \left(-\frac{1}{\varsigma}U'(x)\frac{\partial}{\partial x} + D\frac{\partial^2}{\partial x^2}\right)\phi(x,t).$$

(it is easy to check for the moments in linear potential and harmonic potential as well as the case of free particle).

Rotational diffusion

For planar rotational diffusion the Langevin equations are

$$\begin{aligned} \frac{d\theta}{dt} &= \omega, \\ I\frac{d\omega}{dt} &= -\varsigma\omega + F_\omega(t), \end{aligned}$$

where θ is the angle, ω is the angular velocity, I is the moment of inertia, ς is the friction coefficient and $F(t)$ is a white noise satisfying

$$\begin{aligned} \langle F_\omega(t)\rangle &= 0, \\ \langle F_\omega(t) F_\omega(t)\rangle &= 2kT\varsigma\delta(t-t'). \end{aligned}$$

Since the only energy is the kinetic energy we know that the equilibrium PDF of the angle and angular velocity is

$$f_{eq}(\theta,\omega) = \frac{1}{2\pi}\left(\frac{I}{2\pi kT}\right)^{1/2} e^{-\frac{I}{2kT}\omega^2}.$$

The Fokker-Planck equation for this dynamics is

$$\frac{\partial f}{\partial t} = -\omega\frac{\partial f}{\partial \theta} + \frac{\varsigma kT}{I^2}\frac{\partial}{\partial \omega}\left(\frac{\partial f}{\partial \omega} + \frac{I}{kT}\omega f\right) = \mathcal{D}f.$$

(

$$\begin{aligned} \mathbf{a} &= \begin{pmatrix} \theta \\ \omega \end{pmatrix}; \quad \mathbf{v}(\mathbf{a}) = \begin{pmatrix} \omega \\ -\frac{\varsigma}{I}\omega \end{pmatrix}; \\ F(t) &= \begin{pmatrix} 0 \\ \frac{1}{I}F_\omega(t) \end{pmatrix}; \quad \mathbf{B} = \begin{pmatrix} 0 & 0 \\ 0 & \frac{\varsigma kT}{I^2} \end{pmatrix}; \end{aligned}$$

). One of quantities of interest in this case is the orientation correlation function

$$C_l(t) = \left\langle e^{-il\theta(0)} e^{il\theta(t)} \right\rangle_{eq},$$

where l is an integer. According to our discussion earlier of averages we know that

$$C_l(t) = \int d\theta \int d\omega f_{eq} e^{-il\theta} e^{t\mathcal{D}^+} e^{il\theta} = \int d\theta \int d\omega e^{il\theta} e^{t\mathcal{D}} e^{-il\theta} f_{eq}.$$

Namely, this is equivalent to solving the problem for the initial condition $e^{-il\theta}f_{eq}$. The time dependent PDF must have the same θ dependence as the initial PDF since there is no explicit dependence on θ in the dynamics (there is no potential in θ–preferred direction). Therefore we express the time dependent PDF as

$$f(\theta,\omega,t)=\frac{1}{2\pi}e^{-il\theta}f_l(\omega,t).$$

Then the correlation function is an integral over ω only

$$C_l(t)=\int d\theta\int d\omega e^{il\theta}\frac{1}{2\pi}e^{-il\theta}f_l(\omega,t)=\int d\omega f_l(\omega,t).$$

To solve the FP equation we make the following substitution

$$f_l(\omega,t)=e^{\psi(\omega,t)},$$

so that the FP equation transforms to

$$\begin{aligned}\frac{\partial\psi(\omega,t)}{\partial t} &= il\omega+\frac{\varsigma kT}{I^2}\left(\frac{\partial\psi(\omega,t)}{\partial\omega}\right)^2+\frac{\varsigma}{I}\omega\frac{\partial\psi(\omega,t)}{\partial\omega}\\ &\quad+\frac{\varsigma kT}{I^2}\frac{\partial^2\psi(\omega,t)}{\partial\omega^2}+\frac{\varsigma}{I}.\end{aligned}$$

We use the following ansatz

$$\psi(\omega,t)=a(t)+b(t)\omega-\frac{I}{2kT}\omega^2 \tag{7}$$

which leads to

$$\frac{\partial(a(t)+b(t)\omega)}{\partial t}=il\omega+\frac{\varsigma kT}{I^2}b(t)^2-b(t)\frac{\varsigma}{I}\omega.$$

Collecting terms with equal power of ω we find

$$\begin{aligned}\frac{\partial a(t)}{\partial t} &= \frac{\varsigma kT}{I^2}b(t)^2,\\ \frac{\partial b(t)}{\partial t} &= il-b(t)\frac{\varsigma}{I}.\end{aligned}$$

Since the initial PDF is given by f_{eq} we know that $b(0)=0$ and hence,

$$\begin{aligned}b(t) &= il\frac{I}{\zeta}\left(1-e^{-\zeta t/I}\right)\\ \frac{\partial a}{\partial t} &= -\frac{kT}{\varsigma}l^2\left(1-e^{-\zeta t/I}\right)^2\\ a(t)-a(0) &= -\frac{kT}{\varsigma}l^2\left[t+\frac{1}{2}(-3-e^{-2t\zeta/I}+4e^{-t\zeta/I})\frac{I}{\zeta}\right].\end{aligned}$$

Using the ansatz of equation (7) we find for the correlation function

$$\begin{aligned} C_l(t) &= \int d\omega e^{\psi(\omega,t)} = \int d\omega e^{\left(a(t)-a(0)+b\omega-\frac{I}{2kT}\omega^2\right)} = e^{a(t)-a(0)}e^{\frac{kTb^2}{2I}} \\ &= e^{-\frac{kT}{\varsigma}l^2 t}e^{-\frac{kTI}{\varsigma^2}l^2\left(-1+e^{-t\varsigma/I}\right)} \end{aligned}$$

It is useful to define

$$\begin{aligned} \tau &= \frac{I}{\varsigma} \\ C_l(t) &= e^{-\frac{kTl^2\tau}{I}\left[t-\tau\left(1-e^{-t/\tau}\right)\right]} \end{aligned}$$

The asymptotic behavior is given by

$$C_l(t) \sim \begin{cases} e^{-\frac{kTl^2}{2I}t^2} & t << \tau \\ e^{-\frac{kTl^2\tau}{I}t} & t >> \tau \end{cases} .$$

The Langevin Equation

To understand the Brownian motion more completely, we need to start from the basic physics, i.e., Newton's law of motion. The most direct way of implementing this is to recognize that there is a stochastic component to the force on the particle, which we only know through a probabilistic description. This gives us a Langevin equation for the velocity $u(t)$ (which is a random process)

$$m\frac{du}{dt} = -\gamma u + F(t). \tag{8}$$

Here γu is the deterministic part of the molecular force (the friction), and $F(t)$ is the random component with $\langle F(t)\rangle = 0$ (We could also include a non-stochastic external force, but will not do so here.) We also assume that there is no causal connection of $F(t)$ with the velocity, i.e., $F(t)$ is uncorrelated with the velocity $u(t')$ for $t > t'$. Since the time scale of the molecular collisions is small compared to the time scale m/γ set by the dynamics of the particle, F is a series of randomly spaced spikes or delta functions—a very nasty looking function! However, we are interested in the effect on the time scale m/γ during which many molecular collisions occur, and on this sort of time scale the noisy force behaves as a Gaussian random process implying $\langle F(t)F(t')\rangle = 2B\delta(t-t')$.

The Langevin equation is a complete description (in the stochastic sense!) of the Brownian motion, but is a nasty equation to deal with, since the forcing term is a random sequence of delta functions—very singular! However, we are usually interested in mean values or low order correlation functions, and we can proceed by constructing appropriate quantities and taking expectation values.

Write the equation in the form

$$\frac{du}{dt} = -\frac{u}{\tau} + A(t), \tag{9}$$

where

$$\frac{\gamma}{m} = \frac{1}{\tau}; \frac{F(t)}{m} = A(t).$$

Physically, we expect the stochastic driving to be unaffected by the position and velocity of the particle (remember the average part of the force, which will act in the opposite direction to the particle velocity, is the $\frac{u}{\tau}$ term). Loosely, we would say the force is "uncorrelated with the velocity". However, the velocity responds to the force, so we must be careful:

$$\langle A(t)\, u(t')\rangle = \begin{cases} 0 & t > t' & \text{"force uncorrelated with velocity"} \\ \neq 0 & t < t' & \text{"velocity is correlated with earlier force"} \end{cases}.$$

The formal solution of this equation may be written as

$$u(t) = u(0)\, e^{-\frac{t}{\tau}} + \int_0^t e^{-\frac{t-t'}{\tau}} A(t')\, dt'. \tag{10}$$

Taking the average we find

$$\langle u(t)\rangle = u(0)\, e^{-\frac{t}{\tau}}, \tag{11}$$

because $\langle A(t)\rangle = 0$. For the mean squared velocity we find

$$\begin{aligned} \left\langle u^2(t)\right\rangle &= u^2(0)\, e^{-\frac{2t}{\tau}} + \int_0^t \int_0^t e^{-\frac{t-t'}{\tau}} e^{-\frac{t-t''}{\tau}} \langle A(t')\, A(t'')\rangle\, dt' dt'' \qquad (12) \\ &= u^2(0)\, e^{-\frac{2t}{\tau}} + \int_0^t \int_0^t e^{-\frac{t-t'}{\tau}} e^{-\frac{t-t''}{\tau}} \frac{2B}{m^2} \delta(t'' - t')\, dt' dt'' \\ &= u^2(0)\, e^{-\frac{2t}{\tau}} + \frac{\tau B}{m^2}\left[1 - e^{-\frac{2t}{\tau}}\right]. \end{aligned}$$

In the long time limit, $t \to \infty$, we expect the equipartition theorem to hold, hence

$$\lim_{t\to\infty} \left\langle u^2(t)\right\rangle = \frac{\tau B}{m^2} = \frac{kT}{m} \to B = \frac{m^2 kT}{m\tau} = kT\gamma.$$

This result is a manifestation of the fluctuation dissipation theorem. It implies that the amplitude of the white noise depends on the friction. Such a relation can be expected because the origin of the friction and the white noise is the same–the molecular motions.

The position of the particle, x, is given by the integral over the velocity. To find the position we can multiply the Langevin equation (9) by x.

$$\begin{aligned} xu &= x\frac{dx}{dt} = \frac{1}{2}\frac{dx^2}{dt} \\ x\frac{du}{dt} &= \frac{d\left(xu\right)}{dt} - u^2 = \frac{1}{2}\frac{d^2\left(x^2\right)}{dt^2} - u^2, \\ \frac{1}{2}\frac{d^2\left(x^2\right)}{dt^2} - u^2 &= -\frac{xu}{\tau} + x\left(t\right)A\left(t\right) \end{aligned}$$

Taking the average we find

$$\frac{d^2\left\langle x^2\right\rangle}{dt^2} = -\frac{1}{\tau}\frac{d\left\langle x^2\right\rangle}{dt} + 2\left\langle u^2\right\rangle. \tag{13}$$

For simplicity we can assume that the mean square velocity already takes its long time limit value, $\frac{kT}{m}$. Combining it with a choice of initial conditions $\left\langle x^2\left(t=0\right)\right\rangle = \frac{d\left\langle x^2\right\rangle}{dt}|_{t=0} = 0$, we find

$$\left\langle x^2\left(t\right)\right\rangle = \frac{2kT}{m}\tau^2\left(\frac{t}{\tau} - \left(1 - e^{-t/\tau}\right)\right). \tag{14}$$

For $t << \tau$ we may expand the expression to obtain

$$\left\langle x^2\left(t\right)\right\rangle \sim_{t<<\tau} \frac{kT}{m}t^2, \tag{15}$$

namely, at short times the motion is ballistic with the thermal velocity $\sqrt{\frac{kT}{m}}$. In the other limit of $t >> \tau$, we find

$$\left\langle x^2\left(t\right)\right\rangle \sim_{t>>\tau} \frac{2kT}{m}\tau t, \tag{16}$$

namely, a diffusive motion with the thermal diffusion coefficient $D = \frac{kT}{\gamma}$. Note that here, again, the diffusion coefficient, characterizing the fluctuations, is related to the dissipation.

Langevin equation with mupltiplicative noise

In the previous sections, we considered the Langevin equation with additive noise. Here, we consider the Langevin equation with a multiplicative noise of the form

$$\frac{dy}{dt} = h(y) + C(y)F\left(t\right), \tag{17}$$

with

$$\begin{aligned} \left\langle F\left(t\right)\right\rangle &= 0; \\ \left\langle F\left(t\right)F\left(t\right)\right\rangle &= 2\Gamma\delta\left(t-t'\right). \end{aligned}$$

Naively, one would expect that the transformation of the form

$$\widetilde{y}=\int^{y}\frac{dy'}{C\left(y'\right)};\widetilde{h}\left(\widetilde{y}\right)=\frac{h\left(y\right)}{C\left(y\right)};\widetilde{P}\left(\widetilde{y}\right)=P\left(y\right)C\left(y\right)$$

would result in the following FP equation for the new variables

$$\frac{\partial\widetilde{P}\left(\widetilde{y},t\right)}{\partial t}=-\frac{\partial}{\partial\widetilde{y}}\left(\widetilde{h}\left(\widetilde{y}\right)\widetilde{P}\left(\widetilde{y},t\right)\right)+\Gamma\frac{\partial^{2}}{\partial\widetilde{y}^{2}}\widetilde{P}\left(\widetilde{y},t\right),$$

or transforming back to the original variables

$$\begin{aligned}C\left(y\right)\frac{\partial P\left(y,t\right)}{\partial t} &= -C\left(y\right)\frac{\partial}{\partial y}\left(h\left(y\right)P\left(y,t\right)\right)+\Gamma C\left(y\right)\frac{\partial}{\partial y}\left(C\left(y\right)\frac{\partial}{\partial y}\left(P\left(y,t\right)C\left(y\right)\right)\right),\\ \frac{\partial P\left(y,t\right)}{\partial t} &= -\frac{\partial}{\partial y}\left(h\left(y\right)P\left(y,t\right)\right)+\Gamma\frac{\partial}{\partial y}\left(C\left(y\right)\frac{\partial}{\partial y}\left(P\left(y,t\right)C\left(y\right)\right)\right)\end{aligned}$$

we may also write it as

$$\frac{\partial P\left(y\right)}{\partial t}=-\frac{\partial}{\partial y}\left[\left(h\left(y\right)+\Gamma C\left(y\right)\frac{\partial C\left(y\right)}{\partial y}\right)P\left(y,t\right)\right]+\Gamma\frac{\partial^{2}}{\partial y^{2}}\left[C^{2}\left(y\right)P\left(y,t\right)\right].$$

While everything might seem OK with the above derivation, one has to pay attention to the extra difficulty arising here. The function multiplying the noise, $C(y)$, depends on the value of the dynamic variable, y. However, it is unclear at what time one should take the value, before the "kick" of the force, after the "kick" or considering the average of the value before and after. Therefore, for a noise which is a sequence of "delta" functions, the Langevin equation is not well defined.

In our general derivation of the FP equation, we showed that the general formula for the FP equation is:

$$\frac{\partial P_{2}\left(y,t|y_{0},0\right)}{\partial t}=-\frac{\partial}{\partial y}\left[P_{2}\left(y,t|y_{0},0\right)A\left(y\right)\right]+\frac{1}{2}\frac{\partial^{2}}{\partial y^{2}}\left[P_{2}\left(y,t|y_{0},0\right)B\left(y\right)\right]. \quad (18)$$

where,

$$A\left(y\right) = \lim_{\tau\rightarrow0}\frac{1}{\tau}\int_{-\infty}^{\infty}d\xi\xi P_{2}\left(y+\xi,\tau|y,0\right), \quad (19)$$

$$B\left(y\right) = \lim_{\tau\rightarrow0}\frac{1}{\tau}\int_{-\infty}^{\infty}d\xi\xi^{2}P_{2}\left(y+\xi,\tau|y,0\right). \quad (20)$$

The drift function is a function of the first moment of the displacement (for a given initial position) and the diffusion function is related to the second moment. Using the Langevin equation we can write

$$y(t+\tau)-y\left(t\right)=\xi\left(\tau\right)=\int_{t}^{t+\tau}h(y(t'))dt'+\int_{t}^{t+\tau}C(y(t'))F(t')dt'$$

Now we expand

$$\begin{aligned} h(y(t')) &= h(y(t)) + h'(y(t))\xi(t') + ... \\ C(y(t')) &= C(y(t)) + C'(y(t))\xi(t') + ... \end{aligned}$$

Therefore,

$$\begin{aligned} \xi &= \int_t^{t+\tau} \left[h(y(t)) + h'(y(t))\xi(t'-t)\right] dt' + \int_t^{t+\tau} \left[C(y(t)) + C'(y(t))\xi(t'-t)\right] F(t')dt' \\ &= h(y(t))\tau + h'(y(t)) \int_t^{t+\tau} \xi(t'-t)\, dt' + C(y(t)) \int_t^{t+\tau} F(t')dt' + C'(y(t)) \int_t^{t+\tau} \xi(t'-t)\, F(t')dt' \end{aligned}$$

We expand again $\xi(t'-t)$ in the integrand (neglecting terms which include integrals over the fast decaying function ξ due to the small time interval; the terms we keep involve integrals over functions that do not decay fast);

$$\begin{aligned} \xi(\tau) &= h(y(t))\tau + h'(y(t)) \int_t^{t+\tau} \left[h(y(t))(t'-t) + C(y(t)) \int_t^{t+t'} F(t'')dt'' \right] dt' + C(y(t)) \int_t^{t+\tau} F(t')dt' \\ &\quad + C'(y(t)) \int_t^{t+\tau} \left[h(y(t))\tau + C(y(t)) \int_t^{t+t'} F(t'')dt'' \right] F(t')dt' \end{aligned}$$

Taking the average of both sides and remembering that

$$\begin{aligned} \langle F(t) \rangle &= 0; \\ \langle F(t)\, F(t) \rangle &= 2\Gamma\delta(t-t'). \end{aligned}$$

we find

$$\begin{aligned} \langle \xi(\tau) \rangle &= h(y(t))\tau + h'(y(t))h(y(t))\tau^2/2 + h'(y(t))C(y(t)) \int_t^{t+\tau}\int_t^{t+t'} \langle F(t'') \rangle\, dt''dt' + C(y(t)) \int_t^{t+\tau} \langle F(t') \rangle\, dt' \\ &\quad + C'(y(t))h(y(t)) \int_t^{t+\tau} (t'-t) \langle F(t') \rangle\, dt' + C'(y(t))C(y(t)) \int_t^{t+\tau}\int_t^{t+t'} \langle F(t'')F(t') \rangle\, dt''dt' \\ &= h(y(t))\tau + C'(y(t))C(y(t))2\Gamma \int_0^{\tau}\int_0^{t'} \delta(t''-t')\, dt''dt' \end{aligned}$$

Note that the delta function is centered at the end of the integral. Therefore, it is unclear what is $\int_0^{t'} \delta(t''-t')\, dt''$. It is natural to assume that the delta function is the limit of a sharp peaked symmetric function, i.e.,

$$\delta_\epsilon\left(t-t'\right)=\begin{cases}\frac{1}{\epsilon} & -\frac{\epsilon}{2}<t-t'<\frac{\epsilon}{2}\\ 0 & else\end{cases}.$$

If we take the limit $\epsilon\rightarrow 0$ at the very end we find for every smooth function

$$\int_0^t f\left(t'\right)\delta_\epsilon\left(t-t'\right)dt'=f\left(t\right)/2.$$

This interpretation is equivalent to assuming that

$$\int_t^{t+\tau} C(y(t'))F(t')dt'=\frac{\tau}{2}\left[C(y(t))F(t)+C(y(t+\tau))F(t+\tau)\right].$$

Under this assumption the drift term is

$$A\left(y\right)=\lim_{\tau\rightarrow 0}\frac{1}{\tau}\int_{-\infty}^{\infty}d\xi\xi P_2\left(y+\xi,\tau|y,0\right)=h(y)+C'(y)C(y)\Gamma.$$

This expression implies that for $C'(y)\neq 0$ there is a drift even though the average force is zero. The drift depends on the noise. This interpretation is known as the Stratonovich interpretation. The Ito interpretation assumes that

$$\int_t^{t+\tau} C(y(t'))F(t')dt'=\tau C(y(t))F(t)$$

and therefore the forces in the RHS do not overlap (one is always later than the other) and there is no contribution to the drift from the noise. For the second moment related to the diffusion term both approaches yield

$$\begin{aligned}\left(y(t+\tau)-y\left(t\right)\right)^2 &= \xi^2\left(\tau\right)=\left(\int_t^{t+\tau}A(y(t'))dt'+\int_t^{t+\tau}C(y(t'))F(t')dt'\right)^2\\ &= h^2(y(t))\tau^2+C^2(y(t))\int_t^{t+\tau}\int_t^{t+\tau}F(t')F\left(t''\right)dt''dt'+2h(y)\tau\int_t^{t+\tau}C(y(t'))F(t')dt'+O\left(\tau^2\right)\end{aligned}$$

Taking the average on both sides and remembering the expansion of the integrand in the last equation would result in higher order (in τ) terms we find

$$\left\langle\xi^2\left(\tau\right)\right\rangle=\tau 2\Gamma C^2(y(t))+O\left(\tau^2\right).$$

$$B\left(y\right)=\lim_{\tau\rightarrow 0}\frac{1}{\tau}\int_{-\infty}^{\infty}d\xi\xi^2 P_2\left(y+\xi,\tau|y,0\right)=2\Gamma C^2\left(y\right).$$

Therefore, we see that the two approaches result in different FP equations.

• If $C(y)$ is not constant, the noise term contributes to the drift term $A(y)$: "noise-induced drift", i.e., even if $h(y) = 0$ the average, $\langle y \rangle$, will be time-dependent and is driven purely by the noise term and the dependence of its effect on the state, y.

• The noise-induced drift points to a problem that can arise when one wants to identify the correct Fokker-Planck equation:

- – – Systems with external noise, i.e., the macroscopic dynamics is separate from the noise (one could imagine the noise can be turned off), e.g., a transmission line into which a noisy signal is fed, a bridge under the random force of cars driving on it:

  The macroscopic dynamics, $h(y)$, is known in the absence of noise, and the noise, which conceptually can be turned on or off, can modify the drift term.

  – Systems with internal noise, e.g., Brownian motion, chemical reactions, viscous fluid flow. Here the macroscopic motion arises from the noisy microscopic motion, the noise cannot be turned off. Therefore the macroscopic dynamics $h(y)$ cannot be separated from the noise.

  When the noise affects the system only additively the drift term $A(y)$ is not modified by the noise and the Langevin approach should be fine (viscosity in fluid flow acts only on the linear term, the nonlinear term is the advection term).

  In particular, if the dynamics of the system are also linear the mean satisfies the macroscopic equation.

  When the noise acts nonlinearly then it is not clear what to take for $h(y)$ because $h(y)$ already contains aspects of the noise. E.g., in chemical reactions.

  The nonlinear terms represent reactions of molecules, which are the cause of the noise $\Longrightarrow$ the nonlinear Langevin equation is then most likely not appropriate. One would have to start from the master equation and obtain suitable reductions [van Kampen, Chap. X].

The Ito interpretaion is relevant for external noise for which the action of the noise depends on the value at earlier time while the Startonovic interpretation is relevant for intrinsic noise for which the value of the variable is directly related to the noise.