Gravity 1 - Recitation 12

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Contents

1 Riemann Tensor In 2D

The number of independent components of the Riemann tensor $R_{\rho\sigma\mu\nu}$ is summarized in the following table.

#dimensions	#independent components of Riemann tensor
2	1
3	6
4	20
n	$\frac{n^2\left(n^2-1\right)}{12}$

In two dimensional space the Riemann tensor consists of one independent component, its pure trace part. It is constructed from the Ricci scalar R and the metric. In three dimensional space the Riemann tensor consists of six independent components (as a symmetric second rank tensor does). It is constructed from the Ricci tensor $R_{\mu\nu}$. In four and more dimensional space the Riemann tensor has more independent components, its traceless part (called the Weyl tensor).

Let us focus on two dimensions. $R_{\rho\sigma\mu\nu}$ lives in a one-dimensional vector space of tensors. It can be written as a function times a suitable basis tensor of the same rank and symmetries, build from the metric,

$$R_{\rho\sigma\mu\nu} = \frac{R\left(x^{\mu}\right)}{2}\left(g_{\rho\mu}g_{\sigma\nu} - g_{\rho\nu}g_{\sigma\mu}\right) \tag{1}$$

Decomposition of Riemann tensor in 2-dim.

Also, with index raised

$$R^{\rho}{}_{\sigma\mu\nu} = \frac{R\left(x^{\mu}\right)}{2}\left(\delta^{\rho}_{\mu}g_{\sigma\nu} - \delta^{\rho}_{\nu}g_{\sigma\mu}\right) \tag{2}$$

Why is half the Ricci scalar is the function in front?

The Ricci tensor is

$$R_{\sigma\nu} = g^{\rho\mu}R_{\rho\sigma\mu\nu} = \frac{R}{2}\left(2g_{\sigma\nu} - \delta^{\mu}_{\nu}g_{\sigma\mu}\right) = \frac{R}{2}\left(2g_{\sigma\nu} - g_{\sigma\nu}\right) = \frac{R}{2}g_{\sigma\nu} \tag{3}$$

and two contractions yields

$$g^{\sigma\nu}R_{\sigma\nu} = g^{\sigma\nu}\frac{R}{2}g_{\sigma\nu} = 2\frac{R}{2} = R \tag{4}$$

Remark: In (1) we see that **in two dimensions**, since the Ricci scalar determines the Riemann tensor: $R_{\rho\sigma\mu\nu} = 0$ and the space is flat if and only if

$R=0$; The space is a sphere if and only if R is constant and positive; The space is a hyperboloid if and only if R is constant and negative.

From (1) and (2) we deduce the following short formulas for R

$$R_{1212} = \frac{R}{2}\left(g_{11}g_{22} - g_{12}g_{21}\right) = \frac{Rg}{2} \tag{5}$$

$$R^{1}{}_{212} = \frac{R}{2}\left(\delta^{1}_{1}g_{22} - \delta^{1}_{2}g_{21}\right) = \frac{R}{2}g_{22} \tag{6}$$

Example 1: After the single calculation $R^{\theta}{}_{\phi\theta\phi} = sin^2\theta$ for the sphere metric $ds^2 = a^2\left(d\theta^2 + sin^2\theta d\phi^2\right)$, find R.

$$R = \frac{2R^{\theta}{}_{\phi\theta\phi}}{g_{\phi\phi}} = \frac{2sin^2\theta}{a^2 sin^2\theta} = \frac{2}{a^2} \tag{7}$$

Example 2: Calculate $R^{x}{}_{yxy} = -\left(\frac{h'}{h}\right)'$ for the hyperboloid of radius a in coordinates of the form $ds^2 = h^2\left(y\right)\left(dx^2 + dy^2\right)$ and deduce an ODE for $h\left(y\right)$.

$$R^{x}{}_{yxy} = -\left(\frac{h'}{h}\right)' = \frac{1}{2}\left(-\frac{2}{a^2}\right)h^2 \tag{8}$$

2 Maximally Symmetric Spaces

2.1 Maximal Number Of Symmetries

A *maximally symmetric space* has the same number of symmetries as flat space does. Here symmetry refers to a symmetry of the metric - an isometry.

A flat space in n dimensions has n independent translation symmetries (the number of independent directions to go to) and $\frac{n(n-1)}{2}$ independent rotation symmetries (the number of independent planes to rotate in, $\begin{pmatrix} n \\ 2 \end{pmatrix}$). The total number of symmetries is therefore $n + \frac{n(n-1)}{2} = \frac{n(n+1)}{2}$.

Killing vector fields are the generators of the isometries, so equivalent statement is that **a maximally symmetric space of dimension n possesses $\frac{n(n+1)}{2}$ linearly independent Killing vector fields**.

Let us have a more abstract look on this. It is a consequence of Killing equation and the Bianchi identity that a Killing vector field can be determined by its value ξ_μ and its covatiant derivative $\nabla_\mu \xi_\nu$ at a point. A vector ξ_μ consists of n independent components. Killing equation require that $\nabla_\mu \xi_\nu$ is an antisymmetric tensor, so it consists of $\frac{n(n-1)}{2}$ independent components. This is why the maximal number of independent Killing vector fields is $\frac{n(n+1)}{2}$. A Killing vector field such that $\xi_\mu \neq 0$, $\nabla_\mu \xi_\nu = 0$ at a point, generates a translation of that point. A Killing vector field such that $\xi_\mu = 0$, $\nabla_\mu \xi_\nu \neq 0$ at a point, generates a rotation around that point (the point itself is fixed). Check that for $X^i = (1, 0)$ $Y^i = (0, 1)$ $R^i\,(-y, x)$ at the origin of a flat plane in Cartesian coordinates.

2.2 Constructing A Maximally Symmetric Space

One way to construct an n-dimensional maximally symmetric space is to start one dimension higher, at **flat** $N = n+1$ dimensional space, and to take the n-dimensional hypersurface consisting of all the points at some fixed radius from the origin (the radius of curvature). This is how we constructed the 2-sphere and the 2-hyperboloid. We can do it in any dimension and any signature. The flat embedding space has N translation symmetries and $\frac{N(N-1)}{2}$ rotation symmetries. The n-hypersphere keeps all the rotation symmetries, but only about the origin, and leaves out the translations. Therefore it has $\frac{N(N-1)}{2} = \frac{(n+1)n}{2}$ symmetries, which is maximal for n-dim. space.

For example, consider the 2-sphere. From the extrinsic point of view its three symmetries are the three rotations about the x, y, z axis of $\mathbb{R}^3$. For the intrinsic point of view, observe the north pole. The rotations about the x and y axis are the two translations, moving the point along the two independent directions. The rotation about the z axes keeps the north pole fixed and rotates all points around it, it correspond to the one rotation of a 2-dim. space.

2.3 Curvature Of A Maximally Symmetric Space

The Riemann curvature tensor of an n-dimensional maximally symmetric space has the form

$$R_{\rho\sigma\mu\nu} = \frac{R}{n\,(n-1)}\,(g_{\rho\mu} g_{\sigma\nu} - g_{\rho\nu} g_{\sigma\mu}) \tag{9}$$

Riemann tensor of maximally symmetric space

where R is a **constant** Ricci scalar. Likewise, if $R_{\rho\sigma\mu\nu}$ has the form (9) with constant R then the space is maximally symmetric.

Notice that it has the same form as a general Riemann tensor in two dimensions - but with constant R.

Exercise: Express the Ricci tensor of maximally symmetric space.

$$R_{\sigma\nu} = g^{\rho\mu} R_{\rho\sigma\mu\nu} = \frac{R}{n\left(n-1\right)}\left(n g_{\sigma\nu} - \delta^{\mu}_{\nu} g_{\sigma\mu}\right) = \frac{R}{n\left(n-1\right)}\left(n g_{\sigma\nu} - g_{\sigma\nu}\right) = \frac{1}{n} R g_{\sigma\nu} \tag{10}$$

$$R_{\mu\nu} = \frac{1}{n} R g_{\mu\nu} \tag{11}$$

Ricci tensor of maximally symmetric space

The ricci is proportional to the metric.

3 Robertson - Walker Cosmological Models

3.1 FRW Metric

FRW metrics model an isotropic and homogeneous universe. The spatial three dimensional space is maximally symmetric, while spacetime is not necessarily. such a metric can be written with *comoving coordinates* as

$$ds^2 = -dt^2 + a^2\left(t\right) d\sigma^2 \tag{12}$$

where $d\sigma^2$ is the 3d maximally symmetric spatial metric, and $a\left(t\right)$ is the scale factor.

There are three types of 3d maximally symmetric spatial metrics. Let us use three "spherical" coordinates we know (χ, θ, ϕ).

1. An **Euclidean space** with **zero curvature (flat)**

$$d\sigma_E^2 = d\chi^2 + \chi^2 d\Omega^2 \tag{13}$$

2. A **3-sphere** with **constant positive curvature (closed)**

$$d\sigma_S^2 = d\chi^2 + sin^2\left(\chi\right) d\Omega^2 \tag{14}$$

3. A **3-hyperboloid** with **constant negative curvature (open)**

$$d\sigma_H^2 = d\chi^2 + sinh^2\left(\chi\right) d\Omega^2 \tag{15}$$

where $d\Omega^2 = d\theta^2 + sin^2\left(\theta\right) d\phi^2$ is a metric of a 2-sphere. The 2d maximally symmetric spaces in these coordinates are the $\theta = \frac{\pi}{2}$ or any $\phi = const$ surfaces.

Exercise: Transform these metrics to the common spherically symmetric form with a radial coordinate r

$$d\sigma^2 = f(r)\, dr^2 + r^2 d\Omega^2 \tag{16}$$

and write the FRW metrics (12) in these coordinates.

3.1.1 Euclidean Space

To match between (13) and (16) we require

$$r = \chi \tag{17}$$

Therefore

$$dr = d\chi \tag{18}$$

Plug (17) and (18) into (13) yields

$$d\sigma_E^2 = dr^2 + r^2 d\Omega^2 \tag{19}$$

3.1.2 3-Sphere

To match between (14) and (16) we require

$$r = sin\chi \tag{20}$$

Therefore

$$dr = cos\chi d\chi = \sqrt{1 - sin^2\chi} d\chi = \sqrt{1 - r^2} d\chi \tag{21}$$

$$d\chi^2 = \frac{dr^2}{1 - r^2} \tag{22}$$

Plug (20) and (22) into (14) yields

$$d\sigma_S^2 = \frac{dr^2}{1 - r^2} + r^2 d\Omega^2 \tag{23}$$

3.1.3 3-Hyperboloid

To match between (15) and (16) we require

$$r = sinh\chi \tag{24}$$

Therefore

$$dr = cosh\chi d\chi = \sqrt{1 + sinh^2\chi} d\chi = \sqrt{1 + r^2} d\chi \tag{25}$$

$$d\chi^2 = \frac{dr^2}{1 + r^2} \tag{26}$$

Plug (24) and (26) into (15) yields

$$d\sigma_H^2 = \frac{dr^2}{1 + r^2} + r^2 d\Omega^2 \tag{27}$$

Collecting results (19),(23),(27) into (12), we write the FRW metric in a compact form

FRW metric

$$ds^2 = -dt^2 + a^2(t) \left[\frac{dr^2}{1 - kr^2} + r^2 d\Omega^2 \right] \tag{28}$$

k is the sign of the spatial curvature, $k = 1, 0, -1$ for closed, flat or open universes, respectively.

3.2 Continuity Equation

Exercise: Derive the continuity equation for a perfect fluid in FRW spacetime. Do it by writing the t - component equation of the conservation of the energy-momentum tensor.

The energy-momentum tensor of a perfect fluid is

$$T_{\mu\nu} = \begin{pmatrix} \rho & 0 & 0 & 0 \\ 0 & & & \\ 0 & & pg_{ij} & \\ 0 & & & \end{pmatrix} \qquad T^{\mu}_{\ \nu} = \begin{pmatrix} -\rho & 0 & 0 & 0 \\ 0 & p & 0 & 0 \\ 0 & 0 & p & 0 \\ 0 & 0 & 0 & p \end{pmatrix} \tag{29}$$

The raised index version is simpler.

$$\begin{aligned} 0 = \nabla_\mu T^{\mu}_{\ 0} &= \partial_\mu T^{\mu}_{\ 0} + \Gamma^{\mu}_{\mu\nu} T^{\nu}_{\ 0} - \Gamma^{\nu}_{\mu 0} T^{\mu}_{\ \nu} \\ &= \partial_0 T^{0}_{\ 0} + \Gamma^{\mu}_{\mu 0} T^{0}_{\ 0} - \Gamma^{\nu}_{\mu 0} T^{\mu}_{\ \nu} \\ &= \partial_0 T^{0}_{\ 0} + \Gamma^{i}_{i0} T^{0}_{\ 0} - \Gamma^{i}_{i0} T^{1}_{\ 1} \\ &= -\dot{\rho} - 3\frac{\dot{a}}{a}\rho - 3\frac{\dot{a}}{a} p \end{aligned} \tag{30}$$

Where we used the fact that $T^{\mu}_{\ \nu}$ is diagonal and that $T^{1}_{\ 1} = T^{2}_{\ 2} = T^{3}_{\ 3}$. The only Christofells combination we needed was

$$\Gamma^0_{00} = \frac{1}{2} g^{00} \partial_0 g_{00} = 0 \tag{31}$$

$$\Gamma^i_{i0} = \frac{1}{2} g^{ij} \partial_0 g_{ij} = \frac{1}{2} g^{ij} 2a\dot{a} \frac{g_{ij}}{a^2} = 3\frac{\dot{a}}{a} \tag{32}$$

The continuity equation is

Continuity equation

$$\dot{\rho} + 3\frac{\dot{a}}{a} \left(\rho + p\right) = 0 \tag{33}$$

3.3 Equation Of State And $\rho(a)$

Exercise: Use an equation of state

$$p = w\rho \tag{34}$$

where w is constant, and write the continuity equation (33) with it.

$$\frac{\dot{\rho}}{\rho} = -3\left(1 + w\right) \frac{\dot{a}}{a} \tag{35}$$

Exercise: denote $n \equiv 3\left(w + 1\right)$ and solve for $\rho(a)$.
From (35)

$$\frac{d\rho}{\rho} = -n \frac{da}{a} \tag{36}$$

$$ln\rho = -nlna + C \tag{37}$$

$$\rho(a) = a_0^n \rho_0 a^{-n} \tag{38}$$

where the constant is chosen, for notation purpose, such that $\rho(a_0) = \rho_0$.

3.4 Friedmann Equations And $a(t)$

The Einstein equations for the FRW metric (28) with a perfect fluid source (29) yield the *Friedmann equations*

The (first) Friedmann equation

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi\rho}{3} - \frac{k}{a^2} \tag{39}$$

$$\frac{\ddot{a}}{a} = -\frac{4\pi}{3}\left(\rho + 3p\right) \quad (40)$$

The second Friedmann equation

The second Friedmann equation (40) (with second derivative) is just the *tt*-Einstein equation, while the first Friedmann equation (39) (which is the more famous and useful one) is a combination of the second with the ij - Einstein equations.

For mathematical convenience we treat the curvature term in (39) as kind of energy density with

$$\rho_c \equiv -\frac{3k}{8\pi a^2} \quad (41)$$

and $n = -2$, $w = -\frac{1}{3}$. We rewrite (39) as

$$H^2 = \left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi}{3}\sum_i \rho_i \quad (42)$$

where the sum is on the different kinds of energies, and

$$H = \frac{\dot{a}}{a} \quad (43)$$

is the Hubble parameter.

Exercise: Solve the Friedmann equation, i.e., find $a\left(t\right)$, for one kind of energy density, with $n \neq 0$.

Plug (38) into (42)

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi}{3}\rho = \frac{8\pi}{3}a_0^n \rho_0 a^{-n} \quad (44)$$

$$a^{\frac{n}{2}-1}da = a_0^{\frac{n}{2}}\sqrt{\frac{8\pi}{3}\rho_0}dt \quad (45)$$

$$a\left(t\right) = a_0\left(\frac{n}{2}\sqrt{\frac{8\pi}{3}\rho_0}\right)^{\frac{2}{n}} t^{\frac{2}{n}} \quad (46)$$

The integration constant is chosen to be zero so that $a\left(t = 0\right) = 0$ is the inception of the universe.

	w_i	n_i	$\rho_i(a)$	$a(t)$ for single kind of ρ_i
matter	0	3	$\rho_M \propto a^{-3}$	$a \propto t^{\frac{2}{3}}$
radiation	$\frac{1}{3}$	4	$\rho_R \propto a^{-4}$	$a \propto t^{\frac{1}{2}}$
curvature	$-\frac{1}{3}$	2	$\rho_c \propto a^{-2}$	$a \propto t$
vacuum	-1	0	$\rho_\Lambda \propto a^0$	$a \propto e^{Ht}$

3.5 The Age Of The Universe

Exercise: Calculate the age of the universe with a single kind of source, as a function of w and the Hubble time H_0^{-1} (which can be measured).

The Hubble time is the inverse of the Hubble parameter today

$$H_0^2 = \frac{8\pi}{3}\rho_0 \tag{47}$$

Evaluate (46) at t_0 (today)

$$a_0 = a(t_0) = a_0 \left(\frac{n}{2}\sqrt{\frac{8\pi}{3}\rho_0}\right)^{\frac{2}{n}} t_0^{\frac{2}{n}} \tag{48}$$

$\Rightarrow$

$$1 = \left(\frac{n}{2}\sqrt{\frac{8\pi}{3}\rho_0}\right) t_0 = \frac{n}{2} H_0 t_0 \tag{49}$$

$$\boxed{t_0 = \frac{2}{n} H_0^{-1}} \tag{50}$$

4 Maximally Symmetric Universes

There are three types of maximally symmetric universes.

Minkowski space is a maximally symmetric spacetime with **zero** curvature. A maximally symmetric spacetime with **positive** constant curvature is called *de-Sitter space*. A maximally symmetric spacetime with **negative** constant curvature is called *Anti de-Sitter space* (*AdS*).

4.1 Einstein Vacuum Equation With A Cosmological Constant

Exercise: Write down the vacuum Einstein equation with a cosmological constant Λ, for n dimensional space, without R.

The Einstein equation in vacuum is

$$R_{\mu\nu}-\frac{1}{2}Rg_{\mu\nu}+\Lambda g_{\mu\nu}=0 \tag{51}$$

take the trace (contract with $g^{\mu\nu}$)

$$R-\frac{1}{2}Rn+\Lambda n=0 \tag{52}$$

$\Rightarrow$

$$R=\frac{2n}{n-2}\Lambda \tag{53}$$

Plug (53) back into (51)

$$\boxed{R_{\mu\nu}=\frac{2}{n-2}\Lambda g_{\mu\nu}} \tag{54}$$

Einstein vacuum equation with a cosmological constant

Exercise: Show that a maximally symmetric space is a solution to the vacuum Einstein equation with a cosmological constant, and write its Riemann tensor with Λ.

Plug (53) into (54) the Einstein equation becomes

$$R_{\mu\nu}=\frac{1}{n}Rg_{\mu\nu} \tag{55}$$

We saw in (11) that a maximally symmetric space satisfies this equation, so it is a solution.

For a metric to solve the vacuum Einstein equation **without** a cosmological constant, only its Ricci tensor needs to be zero (such space is called "Ricci-flat"). For a metric to solve the vacuum Einstein equation **with** a cosmological constant, its Ricci tensor needs to have only a pure trace part (55) (such space is called an "Einstein manifold").

We plug (53) into (9) to find the the Riemann tensor

$$R_{\rho\sigma\mu\nu}=\frac{2\Lambda}{(n-1)(n-2)}\left(g_{\rho\mu}g_{\sigma\nu}-g_{\rho\nu}g_{\sigma\mu}\right) \tag{56}$$

Summary: A maximally symmetric space with **positive** constant curvature solves the Einstein vacuum equation with a **positive** cosmological constant. A maximally symmetric space with **negative** constant curvature solves the Einstein vacuum equation with a **negative** cosmological constant.

4.2 De Sitter Space

A maximally symmetric spacetime with **positive** constant curvature is called *de-Sitter space*. We can construct it by considering a 4-dimensional hyperboloid with positive squared radius α^2, embedded in a 5-dimensional flat of signature $(-,+,+,+,+)$ (Minkowski).

With Cartesian coordinates (u,w,x,y,z) for Minkowski space, the 5-dim. metric takes the form

$$ds_5^2 = -du^2 + dw^2 + dx^2 + dy^2 + dz^2 \tag{57}$$

where u is the timelike coordinate. The hyperboloid is the hypersurface satisfying

$$-u^2 + w^2 + x^2 + y^2 + z^2 = \alpha^2 \tag{58}$$

4.2.1 Global Coordinates

Exercise: Find the de-Sitter metric in *global coordinates* (t,χ,θ,ϕ). These coordinates map the 4-dim. hyperboloid as a hyperbola curve of radius α (by coordinate t), where the points on it are 3-shperes of varying radii (with spherical coordinates (χ,θ,ϕ)). [1]

We write (58) as a hyperbola

$$-u^2 + R^2 = \alpha^2 \tag{59}$$

where each point on it is a 3-sphere with radius R

$$w^2 + x^2 + y^2 + z^2 = R^2 \tag{60}$$

Use the radial coordinate R and spherical coordinates of a 3-sphere (χ,θ,ϕ) (just polar-spherical coordinates for the 4-dim. Euclidean space)

$$w = Rcos\chi \tag{61}$$

$$z = Rsin\chi cos\theta \tag{62}$$

$$x = Rsin\chi sin\theta cos\phi \tag{63}$$

$$y = Rsin\chi sin\theta sin\phi \tag{64}$$

[1] Similar to the two coordinates of the 2-hyperboloid that map it as a surface of revolution of hyperbola, where the points on the hyperbola are circles of varying radii.

Then (57) becomes

$$ds_5^2 = -du^2 + dR^2 + R^2 d\Omega_3^2 \tag{65}$$

where the 3-sphere metric is

$$d\Omega_3^2 = d\chi^2 + sin^2\chi \left(d\theta^2 + sin^2\left(\theta\right) d\phi^2\right) \tag{66}$$

Now use t as the hyperbola (59) arc length coordinate ($\frac{t}{\alpha}$ is the hyperbolic angle)

$$u = \alpha sinh\left(\frac{t}{\alpha}\right) \tag{67}$$

$$R = \alpha cosh\left(\frac{t}{\alpha}\right) \tag{68}$$

(65) becomes

$$ds^2 = -dt^2 + R^2 d\Omega_3^2 \tag{69}$$

and the de-Sitter metric in global coordinates is

$$ds^2 = -dt^2 + \alpha^2 cosh^2\left(\frac{t}{\alpha}\right)\left[d\chi^2 + sin^2\chi \left(d\theta^2 + sin^2\left(\theta\right) d\phi^2\right)\right] \tag{70}$$

De Sitter metric in global coordinates

Exercise: Compare the de Sitter metric in global coordinates to the FWR metric (12). What kind of universe does it describe? What is the scale factor $a\left(t\right)$ and what is k?

$$a\left(t\right) = \alpha cosh\left(\frac{t}{\alpha}\right) \tag{71}$$

It describes a closed universe, $k = 1$, that has no beginning. It shrinks to a minimum size at $t = 0$ and then expand again.

4.3 Anti-De Sitter Space

A maximally symmetric spacetime with **negative** constant curvature is called *Anti de-Sitter space (AdS)*. We can construct it by considering a 4-dimensional hyperboloid with negative squared radius $-\alpha^2$, embedded in a 5-dimensional flat space of signature $(-, -, +, +, +)$.

With Cartesian coordinates (u, v, x, y, z) for the flat space space, the 5-dim. metric takes the form

$$ds_5^2 = -du^2 - dv^2 + dx^2 + dy^2 + dz^2 \tag{72}$$

where u and v are timelike coordinates. The hyperboloid is the hypersurface satisfying

$$-u^2 - v^2 + x^2 + y^2 + z^2 = -\alpha^2 \tag{73}$$