כבידה 1

2021/22 סמסטר א'

פתרון תרגיל 6

1. זמן מחזור של מסלול מעגלי

(a) A clock at infinity measures an angular velocity $\Omega = d\phi/dt = \sqrt{M/r^3}$, so the period measured at infinity is:

$$T_\infty = \frac{2\pi}{\Omega} = 2\pi\sqrt{\frac{r^3}{M}} = 2\pi 7^{3/2} M = 116M$$

for $r = 7M$.

(b) To get the period as measured on the spaceship we need:

$$\frac{d\phi}{d\tau} = \frac{d\phi}{dt}\frac{dt}{d\tau} = \Omega\frac{dt}{d\tau}.$$

The 4-velocity u for a timelike circular orbit (r=constant) in the equatorial plane ($\theta = \pi/2$) has components $u = (u^t, 0, 0, u^\phi)$ and mus be normalized to -1:

$$\begin{aligned}
-1 &= u\cdot u = g_{tt}(u^t)^2 + g_{\phi\phi}u^\phi)^2 = -\left(1-\frac{2M}{r}\right)\left(\frac{dt}{d\tau}\right)^2 + r^2\left(\frac{d\phi}{d\tau}\right)^2 \\
&= -\left(1-\frac{2M}{r}\right)\left(\frac{dt}{d\tau}\right)^2 + r^2\Omega^2\left(\frac{dt}{d\tau}\right)^2 \\
&= -\left(1-\frac{2M}{r}-r^2\Omega^2\right)\left(\frac{dt}{d\tau}\right)^2 = -\left(1-\frac{3M}{r}\right)\left(\frac{dt}{d\tau}\right)^2 \\
&= -\frac{4}{7}\left(\frac{dt}{d\tau}\right)^2
\end{aligned}$$

So $dt/d\tau = \sqrt{7/4}$, and:

$$\left(\frac{d\phi}{d\tau}\right) = \sqrt{\frac{7}{4}}\Omega \Rightarrow T_{ship} = \sqrt{\frac{4}{7}}T_\infty \approx 88M.$$

2. יציבות של מסלול מעגלי

$$\begin{aligned} V_{eff} &= -\frac{M}{r}+\frac{l^2}{2r^2}-\frac{Ml^2}{r^2} \\ \frac{\partial V_{eff}}{\partial r} &= 0 \to l^2\left(\frac{3M}{r^4}-\frac{1}{r^3}\right)+\frac{M}{r^2=0} \\ l^2 &= \frac{Mr}{1-\frac{3GM}{r}} \end{aligned}$$

Work out the second derivative of the potential at point r:

$$\begin{aligned} V''(r) &= -\frac{2M}{r^3}+\frac{3l^2}{r^4}-\frac{12Ml^2}{r^5} \\ &= -\frac{2M}{r^3}+\frac{3M}{r^3\left(1-\frac{3M}{r}\right)}-\frac{12\left(M\right)^2}{r^4\left(1-\frac{3M}{r}\right)} \\ &= \frac{M}{r^3}\left(-2+\frac{3}{1-\frac{3M}{r}}-\frac{12M}{r\left(1-\frac{3M}{r}\right)}\right) \\ &= \frac{M}{r^3}\frac{1-\frac{6M}{r}}{1-\frac{3M}{r}}. \end{aligned}$$

This equals ω^2 and so:

$$\omega=\sqrt{\frac{M}{r^3}}\sqrt{\frac{1-\frac{6M}{r}}{1-\frac{3M}{r}}}$$

(b) The condition for closure is that the ratio of period of oscillations to period of rotation should be rational. The period of the orbit was found in Problem 3 and so the condition is:

$$\frac{\tau_{cyc}}{\tau_{pert}}=\sqrt{\frac{r^3}{M}\left(1-\frac{3M}{r}\right)}\sqrt{\frac{M}{r^3}}\sqrt{\frac{1-\frac{6M}{r}}{1-\frac{3M}{r}}}=\sqrt{1-\frac{6M}{r}}$$

This root has to be rational.

(c) Exactly the same as part (a), should give

$$\omega=\sqrt{\frac{M}{r^3}}\sqrt{\frac{\frac{6M}{r}-1}{1-\frac{3M}{r}}}$$

(d) The result is 0 because in this region the potential is indifferent to second order. in fact in this region has no minimum and no maximum but only saddle points, any perturbation will cause the object in the final event to fall into the black hole.

To avoid the suggestive connotations of coordinate names, relabel r and t as follows: $r \to z$, $t \to w$. The metric then takes the form

$$ds^2 = -dw^2 + \frac{4}{9}\left(\frac{(9/2)M}{z-w}\right)^{3/2} dz + \left(\frac{9}{2}M\,(z-w)^2\right)^{2/3} d\Omega^2. \tag{1}$$

Now define a new coordinate, r, by

$$r = \left(\frac{9}{2}M\,(z-w)^2\right)^{1/3}.$$

With r defined this way, we have

$$\begin{aligned} -\sqrt{\frac{2r^3}{9M}} + z &= w \\ dw &= dz - \sqrt{\frac{r}{2M}}dr. \end{aligned}$$

We plug these into the metric and get

$$ds^2 = -\left(1 - \frac{2M}{r}\right) dz^2 + 2\sqrt{\frac{r}{2M}}dzdr - \frac{r}{2M}dr^2 + r^2 d\Omega^2.$$

Next we want to try and diagonalize the dz^2, $dzdr$ and dr^2terms. Define a coordinate t and a function $F(r)$ by

$$z = t + F(r).$$

Substitute this into our new metric:

$$\begin{aligned} ds^2 &= -\left(1 - \frac{2M}{r}\right)\left(dt^2 + 2F'dtdr + F'^2dr^2\right) + \\ &\quad +2\sqrt{\frac{r}{2M}}r\,(dt + F'dr) - \frac{r}{2M}dr^2 + r^2 d\Omega^2 \end{aligned}$$

and choose F so that the metric is diagonal, i.e.

$$\left(1 - \frac{2M}{r}\right) F' = \sqrt{\frac{r}{2M}}.$$

With this choice for F', the metric becomes

$$\begin{aligned} ds^2 &= -\left(1-\frac{2M}{r}\right)dt^2 - \frac{1}{1-\frac{2M}{r}}\left(\frac{r}{2M}\right)dr^2 + \\ &\quad +2\left(\frac{r}{2M}\right)\frac{1}{1-\frac{2M}{r}}dr^2 - \frac{r}{2M}dr^2 + r^2 d\Omega^2 \\ ds^2 &= -\left(\frac{1}{1-\frac{2M}{r}}\right)dt^2 + \frac{1}{1-\frac{2M}{r}}dr^2 + r^2 d\Omega^2 \end{aligned}$$

which you hopefully recognize as the Schwartzschild metric, and is of course a static spacetime.

(b) The coordinates of the first equation above are called "Lemaitre coordinates." The time coordinate w measures the proper time of freely infalling observers; each observer moves along a line z=constant. So a coordinate stationary observer satisfies z=constant, and since $z = t + F(r)$,

$$\begin{aligned} dt &= -F'dr = -\sqrt{\frac{r}{2M}}\frac{1}{1-\frac{2M}{r}}dr \\ \left(\frac{dr}{dt}\right)^2 &= \frac{2M}{r}\left(1-\frac{2M}{r}\right)^2. \end{aligned}$$

Compare this to the expression for w in part (a) (with $u_0 = 1$ for a particle falling from rest at infinity) and you see that the expression here for $\left(\frac{dr}{dt}\right)^2$ represents a radially falling particle, with unit energy-to-rest-mass ratio at infinity.

4. מסע בתוך טיל והסחה לאדום

If the observer O_1doesn't need a rocket, he is free falling. This means he follows a geodesic. The only non trivial geodesic equation is the one for r:

$$\Gamma^r_{tt}\dot{t}^2 + \Gamma^r_{\phi\phi}\dot{\phi}^2 = 0.$$

Then plug in the Christoffel symbols and use

$$\frac{\dot{\phi}}{\dot{t}} = \frac{d\phi}{dt} = \omega$$

:and you get a relation betweeω and R

$$\omega = \frac{m}{R^3}.$$

The possible values of ω- with any power of the rocket engine - have a constraint: the observer has to follow a time like curve $ds^2 < 0$:

$$-\left(1 - \frac{2m}{r}\right) dt^2 + r^2 d\phi^2 < 0$$

and from this you get

$$\omega < \frac{1}{R}\sqrt{\left(1 - \frac{2m}{R}\right)}.$$

Then plug in $\omega = m/R^3$to obtain the possible radii for free falling motion,

$$R > 3m.$$

To find the maximal $|\omega|$, as always for a maximum, you differentiate:

$$\frac{d\omega^2_{max}(R)}{dR} = 0$$

where ω_{max} is the solution which saturates the inequality (makes it equal and not less than). The only solution is

$$R = 3m.$$

To answer part (d) we reformulate and ask what the ratio $\Delta\tau_3/\Delta\tau_2$ of differences in proper time between emitting two photons for O_2 and receiving them for O_3 since the period T of radiation of a given frequency is the wavelength. The situation is static, the coordinate interval $\Delta t_1 = \Delta t_2$for both observers, and

$$d\tau^2 = -\left(1 - \frac{2m}{r}\right) dt^2$$

since they are not moving. Therefore

$$\frac{\lambda'}{\lambda} = \frac{\Delta\tau_2}{\Delta\tau_1} = \sqrt{\frac{1 - 2m/L}{1 - 2m/R}}.$$