פתרון תרגיל 8

1. קואורדינטות אדינגטון – פינקלשטיין

$$t = v - r - 2Mln\left|\frac{r}{2M} - 1\right| \tag{1}$$

For $\frac{r}{2M} - 1 > 0$

$$\begin{aligned} dt &= dv - dr - \left(\frac{r}{2M} - 1\right)^{-1} dr \\ &= dv - \left(1 + \left(\frac{r}{2M} - 1\right)^{-1}\right) dr \\ &= dv - \left(1 + \frac{2M}{r - 2M}\right) dr = dv - \left(\frac{r}{r - 2M}\right) dr \\ &= dv - \left(1 - \frac{2M}{r}\right)^{-1} dr \end{aligned} \tag{2}$$

$$dt^2 = dv^2 - 2\left(1 - \frac{2M}{r}\right)^{-1} dvdr + \left(1 - \frac{2M}{r}\right)^{-2} dr^2 \tag{3}$$

$\Rightarrow$

$$-\left(1 - \frac{2M}{r}\right) dt^2 = -\left(1 - \frac{2M}{r}\right) dv^2 + 2dvdr - \left(1 - \frac{2M}{r}\right)^{-1} dr^2 \tag{4}$$

Plug in Schwarzschild metric in Schwarzschild coordinates

$$ds^2 = -\left(1 - \frac{2M}{r}\right) dt^2 + \left(1 - \frac{2M}{r}\right)^{-1} dr^2 + r^2 d\Omega^2 \tag{5}$$

$$ds^2 = -\left(1 - \frac{2M}{r}\right) dv^2 + 2dvdr + r^2 d\Omega^2 \tag{6}$$

The same hold for $\frac{r}{2M} - 1 < 0$ since $\frac{d}{dr} ln\left(Af\left(r\right)\right) = \frac{d}{dr} ln\left(Bf\left(r\right)\right) = f^{-1}\left(r\right) f'\left(r\right)$ for any constants A, B.

2. זמן מקסימלי לפני פגיעה בסינגולריות

Solution: The path of longest proper time should be a geodesic from $r = 2M$ to $r = 0$. Using (9.26) we have for the elapsed proper time

$$\begin{aligned} \tau &= -\int_{2M}^{0} dr/(dr/d\tau) = -\int_{2M}^{0} dr \left[e^2 - \left(1 + \frac{\ell^2}{r^2}\right)\left(1 - \frac{2M}{r}\right)\right]^{-\frac{1}{2}} \\ &= \int_{0}^{2M} dr \left[e^2 - \left(1 + \frac{\ell^2}{r^2}\right)\left(\frac{2M}{r} - 1\right)\right]^{-\frac{1}{2}} . \end{aligned}$$

Written this way it is clear that non-zero values of ℓ and e only decrease the proper time from $\ell = e = 0$. That geodesic therefore gives the longest time

$$\tau = \int_0^{2M} dr \left(\frac{2M}{r} - 1\right)^{-\frac{1}{2}} = 2M \int_0^1 d\xi \frac{\xi^{\frac{1}{2}}}{\sqrt{1-\xi}} = \pi M .$$

One of the author's students characterized this result as "The more you struggle, the shorter your life."

3. קואורדינטות קרוסקל

Show for $r > 2M$

$$U = \left(\frac{r}{2M} - 1\right)^{\frac{1}{2}} e^{\frac{r}{4M}} cosh\left(\frac{t}{4M}\right)$$
$$V = \left(\frac{r}{2M} - 1\right)^{\frac{1}{2}} e^{\frac{r}{4M}} sinh\left(\frac{t}{4M}\right)$$

Compute the differentials

$$\begin{aligned} dU &= \left[\left(\frac{r}{2M} - 1\right)^{-\frac{1}{2}} + \left(\frac{r}{2M} - 1\right)^{\frac{1}{2}}\right] \frac{1}{4M} e^{\frac{r}{4M}} cosh\left(\frac{t}{4M}\right) dr + \frac{1}{4M}\left(\frac{r}{2M} - 1\right)^{\frac{1}{2}} e^{\frac{r}{4M}} sinh\left(\frac{t}{4M}\right) dt \\ &= \frac{1}{4M} e^{\frac{r}{4M}} \left(\frac{r}{2M} - 1\right)^{\frac{1}{2}} \left[\left(\left(\frac{r}{2M} - 1\right)^{-1} + 1\right) cosh\left(\frac{t}{4M}\right) dr + sinh\left(\frac{t}{4M}\right) dt\right] \\ &= \frac{1}{4M} e^{\frac{r}{4M}} \left(\frac{r}{2M} - 1\right)^{\frac{1}{2}} \left[\left(1 - \frac{2M}{r}\right)^{-1} cosh\left(\frac{t}{4M}\right) dr + sinh\left(\frac{t}{4M}\right) dt\right] \end{aligned} \tag{9}$$

Likewise,

$$dV = \frac{1}{4M} e^{\frac{r}{4M}} \left(\frac{r}{2M} - 1\right)^{\frac{1}{2}} \left[\left(1 - \frac{2M}{r}\right)^{-1} sinh\left(\frac{t}{4M}\right) dr + cosh\left(\frac{t}{4M}\right) dt\right] \tag{10}$$

Therefore

$$dU^2 = \frac{1}{16M^2} e^{\frac{r}{2M}} \left(\frac{r}{2M} - 1\right) \left[\left(1 - \frac{2M}{r}\right)^{-2} cosh^2\left(\frac{t}{4M}\right) dr^2 + sinh^2\left(\frac{t}{4M}\right) dt^2 + 2\left(1 - \frac{2M}{r}\right)^{-1} cosh\left(\frac{t}{4M}\right) sinh\left(\frac{t}{4M}\right) dtdr\right] \tag{11}$$
$$dV^2 = \frac{1}{16M^2} e^{\frac{r}{2M}} \left(\frac{r}{2M} - 1\right) \left[\left(1 - \frac{2M}{r}\right)^{-2} sinh^2\left(\frac{t}{4M}\right) dr^2 + cosh^2\left(\frac{t}{4M}\right) dt^2 + 2\left(1 - \frac{2M}{r}\right)^{-1} sinh\left(\frac{t}{4M}\right) cosh\left(\frac{t}{4M}\right) dtdr\right] \tag{12}$$

$\Rightarrow$

$$\begin{aligned} -dV^2 + dU^2 &= \frac{1}{16M^2} e^{\frac{r}{2M}} \left(\frac{r}{2M} - 1\right) \left[\left(1 - \frac{2M}{r}\right)^{-2} dr^2 - dt^2\right] \\ &= \frac{1}{16M^2} e^{\frac{r}{2M}} \frac{r}{2M} \left(1 - \frac{2M}{r}\right) \left[\left(1 - \frac{2M}{r}\right)^{-2} dr^2 - dt^2\right] \\ &= \frac{r}{32M^3} e^{\frac{r}{2M}} \left[-\left(1 - \frac{2M}{r}\right) dt^2 + \left(1 - \frac{2M}{r}\right)^{-1} dr^2\right] \end{aligned} \tag{13}$$

$\Rightarrow$

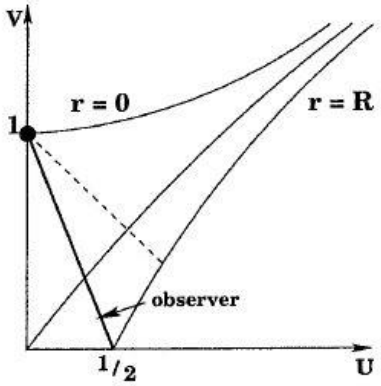
$$\frac{32M^3}{r} e^{-\frac{r}{2M}} \left(-dV^2 + dU^2\right) = -\left(1 - \frac{2M}{r}\right) dt^2 + \left(1 - \frac{2M}{r}\right)^{-1} dr^2 \tag{14}$$

Therefore the Schwarzschild metric in Kruskal coordinates (V, U, θ, ϕ) is

$$ds^2 = \frac{32M^3}{r} e^{-\frac{r}{2M}} \left(-dV^2 + dU^2\right) + r^2\Omega^2 \tag{15}$$

where $r = r\,(V, U)$.

a)



b) The straight line has a slope of 2 which means the observer is within the 45° lines which are the light cones.

c) The latest time is the value of t at which the 45° dotted line from $U = 0$, $V = 1$ intersects the curve $r = R$. The equation of the 45° line is $V = 1 - U$ so,

$$\left(\frac{R}{2M} - 1\right)^{\frac{1}{2}} e^{\frac{R}{4M}} \sinh\left(\frac{t}{4M}\right) = 1 - \left(\frac{R}{2M} - 1\right)^{\frac{1}{2}} e^{\frac{R}{4M}} \cosh\left(\frac{t}{4M}\right)$$

or

$$\frac{1}{2} \sinh\left(\frac{t}{4M}\right) = 1 - \frac{1}{2} \cosh\left(\frac{t}{4M}\right)$$

so that

$$t = 4M \log(2) \ .$$