

Exercise 9 - Solution

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Contents

1	Covariant Derivative Of Vector Fields	2
2	Christoffel Symbols From Metric Compatibility And Torsion-Free Conditions	2
3	Transformation Law For The Christoffel Symbols	3
3.1	From Covariant Derivative Transformation	3
3.2	From Metric Transformation	4
3.3	Christoffels Of Polar Coordinates	5
4	Laplacian	7
4.1	Formula	7
4.2	Cylindrical Coordinates	7
4.3	Spherical Coordinates	7
5	Killing Equation For The Euclidean Plane	9
5.1	In Cartesian Coordinates	9
5.2	Transform To Polar Coordinates	9
5.3	In Polar Coordinates	10

1 Covariant Derivative Of Vector Fields

The first term is the partial derivative, and then for any **upper** index we **add** a Christoffel term, and any **lower** index we **subtract** a Christoffel term.

$$\nabla_q A_{jk} = \partial_q A_{jk} - \Gamma_{qj}^m A_{mk} - \Gamma_{qk}^m A_{jm} \quad (1)$$

$$\nabla_q A^{jk} = \partial_q A^{jk} + \Gamma_{qm}^j A^{mk} + \Gamma_{qm}^k A^{jm} \quad (2)$$

$$\nabla_q A^j_k = \partial_q A^j_k + \Gamma_{qm}^j A^m_k - \Gamma_{qk}^m A^j_m \quad (3)$$

$$\nabla_q A^j_{kl} = \partial_q A^j_{kl} + \Gamma_{qm}^j A^m_{kl} - \Gamma_{qk}^m A^j_{ml} - \Gamma_{ql}^m A^j_{km} \quad (4)$$

$$\nabla_q A^{jkl}_{mn} = \partial_q A^{jkl}_{mn} + \Gamma_{qs}^j A^{skl}_{mn} + \Gamma_{qs}^k A^{jsl}_{mn} + \Gamma_{qs}^l A^{jks}_{mn} - \Gamma_{qm}^s A^{jkl}_{sn} - \Gamma_{qn}^s A^{jkl}_{ms} \quad (5)$$

2 Christoffel Symbols From Metric Compatibility And Torsion-Free Conditions

Write the three cyclic permutations of the metric compatibility condition

$$\nabla_\rho g_{\mu\nu} = \partial_\rho g_{\mu\nu} - \Gamma_{\rho\mu}^\sigma g_{\sigma\nu} - \Gamma_{\rho\nu}^\sigma g_{\mu\sigma} = 0 \quad (6)$$

$$\nabla_\nu g_{\rho\mu} = \partial_\nu g_{\rho\mu} - \Gamma_{\nu\rho}^\sigma g_{\sigma\mu} - \Gamma_{\nu\mu}^\sigma g_{\rho\sigma} = 0 \quad (7)$$

$$\nabla_\mu g_{\nu\rho} = \partial_\mu g_{\nu\rho} - \Gamma_{\mu\nu}^\sigma g_{\sigma\rho} - \Gamma_{\mu\rho}^\sigma g_{\nu\sigma} = 0 \quad (8)$$

Subtract (6)–(7)–(8)

$$\partial_\rho g_{\mu\nu} - \partial_\nu g_{\rho\mu} - \partial_\mu g_{\nu\rho} + 2\Gamma_{\nu\mu}^\sigma g_{\rho\sigma} = 0 \quad (9)$$

where we used that the connection is torsion free $\Gamma_{\mu\nu}^\sigma = \Gamma_{\nu\mu}^\sigma$.

Contract with $g^{\lambda\rho}$

$$\Gamma_{\nu\mu}^\sigma g_{\rho\sigma} g^{\lambda\rho} = \Gamma_{\nu\mu}^\sigma \delta_\sigma^\lambda = \Gamma_{\nu\mu}^\lambda \quad (10)$$

so we get

$$\Gamma_{\nu\mu}^\lambda = \frac{1}{2} g^{\lambda\rho} (\partial_\nu g_{\rho\mu} + \partial_\mu g_{\nu\rho} - \partial_\rho g_{\nu\mu}) \quad (11)$$

3 Transformation Law For The Christoffel Symbols

3.1 From Covariant Derivative Transformation

Transformation of a (1, 1) tensor field

$$\nabla_{i'} V^{j'} = \frac{\partial x^{j'}}{\partial x^j} \frac{\partial x^i}{\partial x^{i'}} \nabla_i V^j \quad (12)$$

$$\frac{\partial}{\partial x^{i'}} V^{j'} + \Gamma_{i'k'}^{j'} V^{k'} = \frac{\partial x^{j'}}{\partial x^j} \frac{\partial x^i}{\partial x^{i'}} \frac{\partial}{\partial x^i} V^j + \frac{\partial x^{j'}}{\partial x^j} \frac{\partial x^i}{\partial x^{i'}} \Gamma_{ik}^j V^k \quad (13)$$

Use the chain rule for the first term of the r.h.s,

$$\frac{\partial x^i}{\partial x^{i'}} \frac{\partial}{\partial x^i} V^j = \frac{\partial}{\partial x^{i'}} V^j \quad (14)$$

(13) becomes

$$\frac{\partial}{\partial x^{i'}} V^{j'} + \Gamma_{i'k'}^{j'} V^{k'} = \frac{\partial x^{j'}}{\partial x^j} \frac{\partial}{\partial x^{i'}} V^j + \frac{\partial x^{j'}}{\partial x^j} \frac{\partial x^i}{\partial x^{i'}} \Gamma_{ik}^j V^k \quad (15)$$

Transform all vectors on the r.h.s to $V^{k'}$

$$V^j = \frac{\partial x^j}{\partial x^{k'}} V^{k'} \quad V^k = \frac{\partial x^k}{\partial x^{k'}} V^{k'} \quad (16)$$

(15) becomes

$$\frac{\partial}{\partial x^{i'}} V^{j'} + \Gamma_{i'k'}^{j'} V^{k'} = \frac{\partial x^{j'}}{\partial x^j} \frac{\partial}{\partial x^{i'}} \left(\frac{\partial x^j}{\partial x^{k'}} V^{k'} \right) + \frac{\partial x^{j'}}{\partial x^j} \frac{\partial x^i}{\partial x^{i'}} \frac{\partial x^k}{\partial x^{k'}} \Gamma_{ik}^j V^{k'} \quad (17)$$

Open the first term on the r.h.s

$$\begin{aligned} \frac{\partial x^{j'}}{\partial x^j} \frac{\partial}{\partial x^{i'}} \left(\frac{\partial x^j}{\partial x^{k'}} V^{k'} \right) &= \frac{\partial x^{j'}}{\partial x^j} \frac{\partial}{\partial x^{i'}} \frac{\partial x^j}{\partial x^{k'}} V^{k'} + \frac{\partial x^{j'}}{\partial x^j} \frac{\partial x^j}{\partial x^{k'}} \frac{\partial}{\partial x^{i'}} V^{k'} \\ &= \frac{\partial x^{j'}}{\partial x^j} \frac{\partial}{\partial x^{i'}} \frac{\partial x^j}{\partial x^{k'}} V^{k'} + \frac{\partial x^{j'}}{\partial x^{k'}} \frac{\partial}{\partial x^{i'}} V^{k'} \\ &= \frac{\partial x^{j'}}{\partial x^j} \frac{\partial}{\partial x^{i'}} \frac{\partial x^j}{\partial x^{k'}} V^{k'} + \delta_{k'}^{j'} \frac{\partial}{\partial x^{i'}} V^{k'} \\ &= \frac{\partial x^{j'}}{\partial x^j} \frac{\partial}{\partial x^{i'}} \frac{\partial x^j}{\partial x^{k'}} V^{k'} + \frac{\partial}{\partial x^{i'}} V^{j'} \end{aligned} \quad (18)$$

(17) becomes

$$\frac{\partial}{\partial x^{i'}} V^{j'} + \Gamma_{i'k'}^{j'} V^{k'} = \frac{\partial x^{j'}}{\partial x^j} \frac{\partial}{\partial x^{i'}} \frac{\partial x^j}{\partial x^{k'}} V^{k'} + \frac{\partial}{\partial x^{i'}} V^{j'} + \frac{\partial x^{j'}}{\partial x^j} \frac{\partial x^i}{\partial x^{i'}} \frac{\partial x^k}{\partial x^{k'}} \Gamma_{ik}^j V^{k'} \quad (19)$$

It is true for any $V^{k'}$, so conclude

$$\Gamma_{i'k'}^{j'} = \frac{\partial x^{j'}}{\partial x^j} \frac{\partial x^i}{\partial x^{i'}} \frac{\partial x^k}{\partial x^{k'}} \Gamma_{ik}^j + \frac{\partial x^{j'}}{\partial x^l} \frac{\partial^2 x^l}{\partial x^{i'} \partial x^{k'}} \quad (20)$$

3.2 From Metric Transformation

The Christoffel symbols are given by the metric as

$$\Gamma_{i'k'}^{j'} = \frac{1}{2} g^{j'm'} \left(\frac{\partial}{\partial x^{i'}} g_{k'm'} + \frac{\partial}{\partial x^{k'}} g_{i'm'} - \frac{\partial}{\partial x^{m'}} g_{i'k'} \right) \quad (21)$$

Transformation of $\frac{\partial}{\partial x^{i'}} g_{k'm'}$

$$\begin{aligned} \frac{\partial}{\partial x^{i'}} g_{k'm'} &= \frac{\partial x^i}{\partial x^{i'}} \frac{\partial}{\partial x^i} \left(\frac{\partial x^k}{\partial x^{k'}} \frac{\partial x^m}{\partial x^{m'}} g_{km} \right) \\ &= \frac{\partial x^i}{\partial x^{i'}} \left(\frac{\partial}{\partial x^i} \frac{\partial x^k}{\partial x^{k'}} \right) \frac{\partial x^m}{\partial x^{m'}} g_{km} + \frac{\partial x^i}{\partial x^{i'}} \frac{\partial x^k}{\partial x^{k'}} \left(\frac{\partial}{\partial x^i} \frac{\partial x^m}{\partial x^{m'}} \right) g_{km} + \frac{\partial x^i}{\partial x^{i'}} \frac{\partial x^k}{\partial x^{k'}} \frac{\partial x^m}{\partial x^{m'}} \frac{\partial}{\partial x^i} g_{km} \\ &= \left(\frac{\partial}{\partial x^{i'}} \frac{\partial x^k}{\partial x^{k'}} \right) \frac{\partial x^m}{\partial x^{m'}} g_{km} + \frac{\partial x^k}{\partial x^{k'}} \left(\frac{\partial}{\partial x^{i'}} \frac{\partial x^m}{\partial x^{m'}} \right) g_{km} + \frac{\partial x^i}{\partial x^{i'}} \frac{\partial x^k}{\partial x^{k'}} \frac{\partial x^m}{\partial x^{m'}} \frac{\partial}{\partial x^i} g_{km} \end{aligned} \quad (22)$$

Likewise,

$$\frac{\partial}{\partial x^{k'}} g_{i'm'} = \left(\frac{\partial}{\partial x^{k'}} \frac{\partial x^i}{\partial x^{i'}} \right) \frac{\partial x^m}{\partial x^{m'}} g_{im} + \frac{\partial x^i}{\partial x^{i'}} \left(\frac{\partial}{\partial x^{k'}} \frac{\partial x^m}{\partial x^{m'}} \right) g_{im} + \frac{\partial x^k}{\partial x^{k'}} \frac{\partial x^i}{\partial x^{i'}} \frac{\partial x^m}{\partial x^{m'}} \frac{\partial}{\partial x^k} g_{im} \quad (23)$$

$$\frac{\partial}{\partial x^{m'}} g_{i'k'} = \left(\frac{\partial}{\partial x^{m'}} \frac{\partial x^i}{\partial x^{i'}} \right) \frac{\partial x^k}{\partial x^{k'}} g_{ik} + \frac{\partial x^i}{\partial x^{i'}} \left(\frac{\partial}{\partial x^{m'}} \frac{\partial x^k}{\partial x^{k'}} \right) g_{ik} + \frac{\partial x^m}{\partial x^{m'}} \frac{\partial x^i}{\partial x^{i'}} \frac{\partial x^k}{\partial x^{k'}} \frac{\partial}{\partial x^m} g_{ik} \quad (24)$$

Use the same dummy indices p, q for all the expressions, and transform the

inverse metric $g^{j'm'} = g^{jm} \frac{\partial x^j}{\partial x^{j'}} \frac{\partial x^m}{\partial x^{m'}}$, the Christoffel (21) is

$$\begin{aligned}
\Gamma_{i'k'}^{j'} &= \\
&= \frac{1}{2} g^{jm} \frac{\partial x^j}{\partial x^{j'}} \frac{\partial x^m}{\partial x^{m'}} \left(\frac{\partial x^p}{\partial x^{m'}} \frac{\partial}{\partial x^{i'}} \frac{\partial x^q}{\partial x^{k'}} + \frac{\partial x^p}{\partial x^{k'}} \frac{\partial}{\partial x^{i'}} \frac{\partial x^q}{\partial x^{m'}} \right) g_{pq} \\
&+ \frac{1}{2} g^{jm} \frac{\partial x^j}{\partial x^{j'}} \frac{\partial x^m}{\partial x^{m'}} \left(\frac{\partial x^p}{\partial x^{m'}} \frac{\partial}{\partial x^{k'}} \frac{\partial x^q}{\partial x^{i'}} + \frac{\partial x^p}{\partial x^{i'}} \frac{\partial}{\partial x^{k'}} \frac{\partial x^q}{\partial x^{m'}} \right) g_{pq} \\
&- \frac{1}{2} g^{jm} \frac{\partial x^j}{\partial x^{j'}} \frac{\partial x^m}{\partial x^{m'}} \left(\frac{\partial x^p}{\partial x^{k'}} \frac{\partial}{\partial x^{m'}} \frac{\partial x^q}{\partial x^{i'}} + \frac{\partial x^p}{\partial x^{i'}} \frac{\partial}{\partial x^{m'}} \frac{\partial x^q}{\partial x^{k'}} \right) g_{pq} \\
&+ \frac{1}{2} g^{jn} \frac{\partial x^j}{\partial x^{j'}} \frac{\partial x^m}{\partial x^{m'}} \frac{\partial x^i}{\partial x^{i'}} \frac{\partial x^k}{\partial x^{k'}} \frac{\partial x^n}{\partial x^{n'}} \left(\frac{\partial}{\partial x^i} g_{km} + \frac{\partial}{\partial x^k} g_{im} - \frac{\partial}{\partial x^m} g_{ik} \right) \\
&= g^{jm} \frac{\partial x^j}{\partial x^{j'}} \delta_m^p \frac{\partial}{\partial x^{i'}} \frac{\partial x^q}{\partial x^{k'}} g_{pq} + \frac{1}{2} g^{jn} \delta_n^m \frac{\partial x^j}{\partial x^{j'}} \frac{\partial x^i}{\partial x^{i'}} \frac{\partial x^k}{\partial x^{k'}} \left(\frac{\partial}{\partial x^i} g_{km} + \frac{\partial}{\partial x^k} g_{im} - \frac{\partial}{\partial x^m} g_{ik} \right) \\
&= \frac{\partial x^j}{\partial x^{j'}} \frac{\partial}{\partial x^{i'}} \frac{\partial x^q}{\partial x^{k'}} + \frac{\partial x^j}{\partial x^{j'}} \frac{\partial x^i}{\partial x^{i'}} \frac{\partial x^k}{\partial x^{k'}} \frac{1}{2} g^{jm} \left(\frac{\partial}{\partial x^i} g_{km} + \frac{\partial}{\partial x^k} g_{im} - \frac{\partial}{\partial x^m} g_{ik} \right) \\
&= \frac{\partial x^j}{\partial x^{j'}} \frac{\partial x^i}{\partial x^{i'}} \frac{\partial x^k}{\partial x^{k'}} \Gamma_{ik}^j + \frac{\partial x^j}{\partial x^{j'}} \frac{\partial^2 x^q}{\partial x^{i'} \partial x^{k'}} \tag{25}
\end{aligned}$$

where we used things like $\frac{\partial x^m}{\partial x^n} \frac{\partial x^m}{\partial x^{m'}} = \frac{\partial x^m}{\partial x^n} = \delta_n^m$, $g^{jm} \delta_m^p g_{pq} = g^{jp} g_{pq} = \delta_q^j$, ecc., and that partial derivatives commute.

3.3 Christoffels Of Polar Coordinates

Start with Cartesian coordinates ($x^1 = x, x^2 = y$), where all $\Gamma_{ik}^j = 0$.

By (20), we need to calculate

$$\Gamma_{i'k'}^{j'} = \frac{\partial x^j}{\partial x^{j'}} \frac{\partial^2 x^l}{\partial x^{i'} \partial x^{k'}} = [J^{-1} \partial_{i'} J]_{k'}^{j'} \tag{26}$$

Polar coordinates are ($x^1 = r, x^2 = \theta$)

$$x = r \cos \theta \quad y = r \sin \theta \tag{27}$$

Compute the partial derivatives, the Jacobian matrix is:

$$J_{k'}^l = \frac{\partial x^l}{\partial x^{k'}} = \begin{pmatrix} \cos \theta & -r \sin \theta \\ \sin \theta & r \cos \theta \end{pmatrix} \tag{28}$$

The inverse transformation is

$$r^2 = x^2 + y^2 \quad \tan\theta = \frac{y}{x} \quad (29)$$

Compute the partial derivatives:

$$r \frac{\partial r}{\partial x} = x \Rightarrow \frac{\partial r}{\partial x} = \frac{x}{r} = \cos\theta \quad (30)$$

$$r \frac{\partial r}{\partial y} = y \Rightarrow \frac{\partial r}{\partial y} = \frac{y}{r} = \sin\theta \quad (31)$$

$$\frac{1}{\cos^2\theta} \frac{\partial\theta}{\partial x} = -\frac{y}{x^2} \Rightarrow \frac{\partial\theta}{\partial x} = -\frac{y\cos^2\theta}{x^2} = -\frac{\sin\theta}{r} \quad (32)$$

$$\frac{1}{\cos^2\theta} \frac{\partial\theta}{\partial y} = \frac{1}{x} \Rightarrow \frac{\partial\theta}{\partial y} = \frac{\cos^2\theta}{x} = \frac{\cos\theta}{r} \quad (33)$$

The inverse Jacobian matrix is

$$J_{l'}^{k'} = \frac{\partial x^{k'}}{\partial x^{l'}} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\frac{\sin\theta}{r} & \frac{\cos\theta}{r} \end{pmatrix} \quad (34)$$

Calculate Christoffels $\Gamma_{rk'}^{j'}$ by plugin (28) and (34) into (26) with $i' = r$

$$\begin{aligned} \Gamma_{rk'}^{j'} &= \begin{pmatrix} \cos\theta & \sin\theta \\ -\frac{\sin\theta}{r} & \frac{\cos\theta}{r} \end{pmatrix} \frac{\partial}{\partial r} \begin{pmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{pmatrix} \\ &= \begin{pmatrix} \cos\theta & \sin\theta \\ -\frac{\sin\theta}{r} & \frac{\cos\theta}{r} \end{pmatrix} \begin{pmatrix} 0 & -\sin\theta \\ 0 & \cos\theta \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & \frac{\sin^2\theta}{r} + \frac{\cos^2\theta}{r} \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & \frac{1}{r} \end{pmatrix} \end{aligned} \quad (35)$$

The non-vanishing Christoffel here is

$$\Gamma_{r\theta}^\theta = \frac{1}{r} \quad (36)$$

Calculate Christoffels $\Gamma_{\theta k'}^{j'}$ by plugin (28) and (34) into (26) with $i' = \theta$

$$\begin{aligned} \Gamma_{\theta k'}^{j'} &= \begin{pmatrix} \cos\theta & \sin\theta \\ -\frac{\sin\theta}{r} & \frac{\cos\theta}{r} \end{pmatrix} \frac{\partial}{\partial\theta} \begin{pmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{pmatrix} \\ &= \begin{pmatrix} \cos\theta & \sin\theta \\ -\frac{\sin\theta}{r} & \frac{\cos\theta}{r} \end{pmatrix} \begin{pmatrix} -\sin\theta & -r\cos\theta \\ \cos\theta & -r\sin\theta \end{pmatrix} = \begin{pmatrix} 0 & -r \\ \frac{1}{r} & 0 \end{pmatrix} \end{aligned}$$

The remaining non-vanishing Christoffels are

$$\Gamma_{\theta r}^{\theta} = \frac{1}{r} \quad \Gamma_{\theta\theta}^r = -r \quad (37)$$

You can also calculate each component separately, without use of matrices.

4 Laplacian

4.1 Formula

See recitation 9 section 2.3.

$$\nabla^2 f = \frac{1}{\sqrt{g}} \partial_i (\sqrt{g} g^{ij} \partial_j f) \quad (38)$$

4.2 Cylindrical Coordinates

The metric in cylindrical coordinates (ρ, θ, z) is

$$ds^2 = d\rho^2 + \rho^2 d\theta^2 + dz^2 \quad (39)$$

$$g_{ij} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \rho^2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad g^{ij} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{\rho^2} & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad (40)$$

$$g = \det(g_{ij}) = \rho^2 \quad (41)$$

$$\sqrt{g} = \rho \quad (42)$$

Notice that the metric is diagonal.

$$\nabla^2 f = \frac{1}{\rho} \partial_{\rho} (\rho \partial_{\rho} f) + \frac{1}{\rho} \partial_{\theta} \left(\rho \frac{1}{\rho^2} \partial_{\theta} f \right) + \frac{1}{\rho} \partial_z (\rho \partial_z f) \quad (43)$$

$$\nabla^2 f = \frac{1}{\rho} \frac{\partial}{\partial \rho} \left(\rho \frac{\partial f}{\partial \rho} \right) + \frac{1}{\rho^2} \frac{\partial^2 f}{\partial \theta^2} + \frac{\partial f}{\partial z^2} \quad (44)$$

4.3 Spherical Coordinates

The metric in spherical coordinates (r, θ, ϕ) is

$$ds^2 = dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2 \quad (45)$$

$$g_{ij} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & r^2 \sin^2 \theta \end{pmatrix} \quad g^{ij} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{r^2} & 0 \\ 0 & 0 & \frac{1}{r^2 \sin^2 \theta} \end{pmatrix} \quad (46)$$

$$g = \det(g_{ij}) = r^4 \sin^2 \theta \quad (47)$$

$$\sqrt{g} = r^2 \sin \theta \quad (48)$$

Notice that the metric is diagonal.

The short way: Use metric determinant and inverse metric

$$\nabla^2 f = \frac{1}{r^2 \sin \theta} \partial_r (r^2 \sin \theta \partial_r f) + \frac{1}{r^2 \sin \theta} \partial_\theta \left(r^2 \sin \theta \frac{1}{r^2} \partial_\theta f \right) + \frac{1}{r^2 \sin \theta} \partial_\phi \left(r^2 \sin \theta \frac{1}{r^2 \sin^2 \theta} \partial_\phi f \right) \quad (49)$$

$$\nabla^2 f = \frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial f}{\partial r} \right) + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \left(\sin \theta \frac{\partial f}{\partial \theta} \right) + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2 f}{\partial \phi^2} \quad (50)$$

The long and winding road:

Use Christoffels (extra work that is skipped here)

$$\Gamma_{\theta\theta}^r = -r \quad \Gamma_{\theta r}^\theta = \frac{1}{r} \quad \Gamma_{\phi\phi}^r = -r \sin^2 \theta \quad \Gamma_{\phi r}^\phi = \frac{1}{r} \quad (51)$$

$$\Gamma_{\phi\phi}^\theta = -\sin \theta \cos \theta \quad \Gamma_{\phi\theta}^\phi = \cot \theta \quad (52)$$

$$\begin{aligned} g^{ij} \nabla_i \nabla_j f &= \nabla_r \partial_r f + \frac{1}{r^2} \nabla_\theta \partial_\theta f + \frac{1}{r^2 \sin^2 \theta} \nabla_\phi \partial_\phi f \\ &= \partial_r \partial_r f - \Gamma_{rr}^k \partial_k f + \frac{1}{r^2} (\partial_\theta \partial_\theta f - \Gamma_{\theta\theta}^k \partial_k f) + \frac{1}{r^2 \sin^2 \theta} (\partial_\phi \partial_\phi f - \Gamma_{\phi\phi}^k \partial_k f) \\ &= \partial_r \partial_r f + \frac{1}{r^2} \partial_\theta \partial_\theta f - \frac{1}{r^2} \Gamma_{\theta\theta}^r \partial_r f + \frac{1}{r^2 \sin^2 \theta} \partial_\phi \partial_\phi f - \frac{1}{r^2 \sin^2 \theta} \Gamma_{\phi\phi}^r \partial_r f - \frac{1}{r^2 \sin^2 \theta} \Gamma_{\phi\phi}^\theta \partial_\theta f \\ &= \partial_r \partial_r f + \frac{1}{r^2} \partial_\theta \partial_\theta f + \frac{1}{r} \partial_r f + \frac{1}{r^2 \sin^2 \theta} \partial_\phi \partial_\phi f + \frac{1}{r} \partial_r f + \frac{1}{r^2 \sin \theta} \cos \theta \partial_\theta f \\ &= \partial_r \partial_r f + \frac{2}{r} \partial_r f + \frac{1}{r^2} \partial_\theta \partial_\theta f + \frac{1}{r^2 \sin \theta} (\partial_\theta \sin \theta) \partial_\theta f + \frac{1}{r^2 \sin^2 \theta} \partial_\phi \partial_\phi f \\ &= \partial_r \partial_r f + \frac{1}{r^2} (\partial_r r^2) \partial_r f + \frac{1}{r^2} \partial_\theta \partial_\theta f + \frac{1}{r^2 \sin \theta} (\partial_\theta \sin \theta) \partial_\theta f + \frac{1}{r^2 \sin^2 \theta} \partial_\phi \partial_\phi f \\ &= \frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial f}{\partial r} \right) + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \left(\sin \theta \frac{\partial f}{\partial \theta} \right) + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2 f}{\partial \phi^2} \quad (53) \end{aligned}$$

We got the same result!

5 Killing Equation For The Euclidean Plane

5.1 In Cartesian Coordinates

In Cartesian coordinates $g_{ij} = \delta_{ij}$ and all $\Gamma_{jk}^i = 0$. So there is no difference between upper and lower indices, and covariant derivatives reduce to partial derivatives. Therefore, Killing equation takes the form

$$\partial_i \xi^j + \partial_j \xi^i = 0 \quad (54)$$

Clearly for both

$$X^i = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad Y^i = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad (55)$$

$$\begin{bmatrix} 2\partial_x \xi^x & \partial_x \xi^y + \partial_y \xi^x \\ \partial_y \xi^x + \partial_x \xi^y & 2\partial_y \xi^y \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \quad (56)$$

For

$$R^i = \begin{pmatrix} -y \\ x \end{pmatrix} \quad (57)$$

$$\begin{bmatrix} 2\partial_x R^x & \partial_x R^y + \partial_y R^x \\ " & 2\partial_y R^y \end{bmatrix} = \begin{bmatrix} 0 & 1-1 \\ " & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \quad (58)$$

Remark: xy and yx components are identical so there is no need to calculate them both, we wrote " to indicate it is the same as the other component.

5.2 Transform To Polar Coordinates

We already found the matrices (28)

$$\frac{\partial x^i}{\partial x^{k'}} = \begin{pmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{pmatrix} \quad (59)$$

and (34)

$$\frac{\partial x^{k'}}{\partial x^j} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\frac{\sin\theta}{r} & \frac{\cos\theta}{r} \end{pmatrix} \quad (60)$$

Multiply

$$\frac{\partial x^i}{\partial x^{k'}} \frac{\partial x^{k'}}{\partial x^j} = \begin{pmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{pmatrix} \begin{pmatrix} \cos\theta & \sin\theta \\ -\frac{\sin\theta}{r} & \frac{\cos\theta}{r} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \delta_j^i \quad (61)$$

Vector transformation:

$$X^{i'} = \frac{\partial x^{i'}}{\partial x^i} X^i = \begin{pmatrix} \cos\theta & \sin\theta \\ -\frac{\sin\theta}{r} & \frac{\cos\theta}{r} \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} \cos\theta \\ -\frac{\sin\theta}{r} \end{pmatrix} \quad (62)$$

$$Y^{i'} = \frac{\partial x^{i'}}{\partial x^i} Y^i = \begin{pmatrix} \cos\theta & \sin\theta \\ -\frac{\sin\theta}{r} & \frac{\cos\theta}{r} \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} \sin\theta \\ \frac{\cos\theta}{r} \end{pmatrix} \quad (63)$$

$$R^{i'} = \frac{\partial x^{i'}}{\partial x^i} R^i = \begin{pmatrix} \cos\theta & \sin\theta \\ -\frac{\sin\theta}{r} & \frac{\cos\theta}{r} \end{pmatrix} \begin{pmatrix} -r\sin\theta \\ r\cos\theta \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad (64)$$

5.3 In Polar Coordinates

To write Killing equation in polar coordinates we need to lower the indices of the vector fields, and use the covariant derivatives. The metric is

$$g_{i'j'} = \begin{pmatrix} 1 & 0 \\ 0 & r^2 \end{pmatrix} \quad (65)$$

$$X_{i'} = g_{i'j'} X^{j'} = \begin{pmatrix} X^r & r^2 X^\theta \end{pmatrix} = \begin{pmatrix} \cos\theta & -r\sin\theta \end{pmatrix} \quad (66)$$

$$Y_{i'} = g_{i'j'} Y^{j'} = \begin{pmatrix} Y^r & r^2 Y^\theta \end{pmatrix} = \begin{pmatrix} \sin\theta & r\cos\theta \end{pmatrix} \quad (67)$$

$$R_{i'} = g_{i'j'} R^{j'} = \begin{pmatrix} R^r & r^2 R^\theta \end{pmatrix} = \begin{pmatrix} 0 & r^2 \end{pmatrix} \quad (68)$$

We already calculated the Christoffels in (36) and (37).

Killing equation is

$$\nabla_{i'} \xi_{j'} + \nabla_{j'} \xi_{i'} = 0 \quad (69)$$

Its components are

$$0 = \nabla_r \xi_r = \partial_r \xi_r - \Gamma_{rr}^{k'} \xi_{k'} = \partial_r \xi_r \quad (70)$$

$$0 = \nabla_\theta \xi_\theta = \partial_\theta \xi_\theta - \Gamma_{\theta\theta}^{k'} \xi_{k'} = \partial_\theta \xi_\theta - \Gamma_{\theta\theta}^r \xi_r = \partial_\theta \xi_\theta + r \xi_r \quad (71)$$

$$\begin{aligned}
0 &= \nabla_r \xi_\theta + \nabla_\theta \xi_r = \partial_r \xi_\theta - \Gamma_{r\theta}^{k'} \xi_{k'} + \partial_\theta \xi_r - \Gamma_{\theta r}^{k'} \xi_{k'} = \partial_r \xi_\theta - \Gamma_{r\theta}^{k'} \xi_{k'} + \partial_\theta \xi_r - \Gamma_{\theta r}^{k'} \xi_{k'} \\
&= \partial_r \xi_\theta - \Gamma_{r\theta}^\theta \xi_\theta + \partial_\theta \xi_r - \Gamma_{\theta r}^\theta \xi_\theta = \partial_r \xi_\theta + \partial_\theta \xi_r - \frac{2}{r} \xi_\theta
\end{aligned} \tag{72}$$

Check for $X_{i'} = (\cos\theta \quad -r\sin\theta)$:

$$\partial_r X_r = 0 \tag{73}$$

$$\partial_\theta X_\theta + r X_r = -r\cos\theta + r\cos\theta = 0 \tag{74}$$

$$\partial_r X_\theta + \partial_\theta X_r - \frac{2}{r} X_\theta = -\sin\theta - \sin\theta - \frac{2}{r} (-r\sin\theta) = 0 \tag{75}$$

Check for $Y_{i'} = (\sin\theta \quad r\cos\theta)$:

$$\partial_r Y_r = 0 \tag{76}$$

$$\partial_\theta Y_\theta + r Y_r = -r\sin\theta + r\sin\theta = 0 \tag{77}$$

$$\partial_r Y_\theta + \partial_\theta Y_r - \frac{2}{r} Y_\theta = \cos\theta + \cos\theta - \frac{2}{r} (r\cos\theta) = 0 \tag{78}$$

Check for $R_{i'} = (0 \quad r^2)$:

$$\partial_r R_r = 0 \tag{79}$$

$$\partial_\theta R_\theta + r R_r = 0 + 0 = 0 \tag{80}$$

$$\partial_r R_\theta + \partial_\theta R_r - \frac{2}{r} R_\theta = 2r + 0 - \frac{2}{r} (r^2) = 0 \tag{81}$$

X is the Killing vector field that generates the translation symmetry in the x-direction, Y is the Killing vector field that generates the translation symmetry in the y-direction, and R is the Killing vector field that generates the rotation symmetry about the plane origin. These are three linearly independent Killing vector fields, which is the maximal number of isometries for a 2-dimensional space, i.e., the Euclidean plane is a maximally symmetric space.