

Exercise 10 - Solution

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Contents

1	Riemann And Ricci Tensors Identities	2
1.1	Riemann Tensor In Local Inertial Coordinates	2
1.2	Antisymmetry In First Two Indices	2
1.3	Antisymmetry In Last Two Indices	3
1.4	Symmetry In Exchanging Pairs Of Indices	3
1.5	Cyclic Identity	3
1.6	Bianchi Identity	3
1.7	Ricci Tensor Is Symmetric	4
1.8	Contractions Of Riemann	4
1.9	Contracted Bianchi Identity	5
2	The Hyperbolic Plane	5
2.1	The Poincare Half Plane	5
2.2	The Poincare Disc	7
3	A Uniform Gravitational Field	9
4	Curvature Of The Schwarzschild Metric	10
4.1	Riemann Tensor	10
4.2	Ricci Scalar	10
4.3	Some Scalars	10
4.4	Divergence	10
5	Geodesic Deviation Equation	11

1 Riemann And Ricci Tensors Identities

1.1 Riemann Tensor In Local Inertial Coordinates

In local inertial coordinates, at point x

$$g_{\mu\nu}(x) = \eta_{\mu\nu} \quad \partial_\rho g_{\mu\nu}(x) = 0 \quad \Gamma_{\mu\nu}^\rho(x) = 0 \quad \partial_\rho \partial_\sigma g_{\mu\nu} \neq 0 \quad (1)$$

thus Riemann tensor contains only derivatives of Γ

$$\begin{aligned} R_{\rho\sigma\mu\nu} &= g_{\rho\lambda} R^\lambda_{\sigma\mu\nu} = g_{\rho\lambda} (\partial_\mu \Gamma^\lambda_{\sigma\nu} - \partial_\nu \Gamma^\lambda_{\sigma\mu}) \\ &= g_{\rho\lambda} \partial_\mu \left(\frac{1}{2} g^{\lambda\alpha} (\partial_\sigma g_{\nu\alpha} + \partial_\nu g_{\sigma\alpha} - \partial_\alpha g_{\sigma\nu}) \right) - g_{\rho\lambda} \partial_\nu \left(\frac{1}{2} g^{\lambda\alpha} (\partial_\sigma g_{\mu\alpha} + \partial_\mu g_{\sigma\alpha} - \partial_\alpha g_{\sigma\mu}) \right) \\ &= \frac{1}{2} g_{\rho\lambda} g^{\lambda\alpha} \partial_\mu (\partial_\sigma g_{\nu\alpha} + \partial_\nu g_{\sigma\alpha} - \partial_\alpha g_{\sigma\nu}) - \frac{1}{2} g_{\rho\lambda} g^{\lambda\alpha} \partial_\nu (\partial_\sigma g_{\mu\alpha} + \partial_\mu g_{\sigma\alpha} - \partial_\alpha g_{\sigma\mu}) \\ &= \frac{1}{2} \delta_\rho^\alpha [(\partial_\mu \partial_\sigma g_{\nu\alpha} + \cancel{\partial_\mu \partial_\nu g_{\sigma\alpha}} - \partial_\mu \partial_\alpha g_{\sigma\nu}) - (\partial_\nu \partial_\sigma g_{\mu\alpha} + \cancel{\partial_\nu \partial_\mu g_{\sigma\alpha}} - \partial_\nu \partial_\alpha g_{\sigma\mu})] \\ &= \frac{1}{2} (\partial_\mu \partial_\sigma g_{\nu\rho} - \partial_\mu \partial_\rho g_{\sigma\nu} - \partial_\nu \partial_\sigma g_{\mu\rho} + \partial_\nu \partial_\rho g_{\sigma\mu}) \end{aligned} \quad (2)$$

$$R_{\rho\sigma\mu\nu} = \frac{1}{2} (\partial_\mu \partial_\sigma g_{\nu\rho} - \partial_\mu \partial_\rho g_{\sigma\nu} - \partial_\nu \partial_\sigma g_{\mu\rho} + \partial_\nu \partial_\rho g_{\sigma\mu})$$

The following tensor identities we can prove in these coordinates, and the equations will hold in any coordinates.

1.2 Antisymmetry In First Two Indices

$$\begin{aligned} R_{\rho\sigma\mu\nu} &= \frac{1}{2} (\partial_\mu \partial_\sigma g_{\nu\rho} - \partial_\mu \partial_\rho g_{\sigma\nu} - \partial_\nu \partial_\sigma g_{\mu\rho} + \partial_\nu \partial_\rho g_{\sigma\mu}) \\ &= -\frac{1}{2} (-\partial_\mu \partial_\sigma g_{\nu\rho} + \partial_\mu \partial_\rho g_{\sigma\nu} + \partial_\nu \partial_\sigma g_{\mu\rho} - \partial_\nu \partial_\rho g_{\sigma\mu}) \\ &= -\frac{1}{2} (\partial_\mu \partial_\rho g_{\sigma\nu} - \partial_\mu \partial_\sigma g_{\nu\rho} - \partial_\nu \partial_\rho g_{\sigma\mu} + \partial_\nu \partial_\sigma g_{\mu\rho}) \\ &= -R_{\sigma\rho\mu\nu} \end{aligned} \quad (3)$$

1.3 Antisymmetry In Last Two Indices

$$\begin{aligned}
R_{\rho\sigma\mu\nu} &= \frac{1}{2} (\partial_\mu \partial_\sigma g_{\nu\rho} - \partial_\mu \partial_\rho g_{\sigma\nu} - \partial_\nu \partial_\sigma g_{\mu\rho} + \partial_\nu \partial_\rho g_{\sigma\mu}) \\
&= -\frac{1}{2} (-\partial_\mu \partial_\sigma g_{\nu\rho} + \partial_\mu \partial_\rho g_{\sigma\nu} + \partial_\nu \partial_\sigma g_{\mu\rho} - \partial_\nu \partial_\rho g_{\sigma\mu}) \\
&= -\frac{1}{2} (\partial_\nu \partial_\sigma g_{\mu\rho} - \partial_\nu \partial_\rho g_{\sigma\mu} - \partial_\mu \partial_\sigma g_{\nu\rho} + \partial_\mu \partial_\rho g_{\sigma\nu}) \\
&= -R_{\rho\sigma\nu\mu}
\end{aligned} \tag{4}$$

1.4 Symmetry In Exchanging Pairs Of Indices

$$\begin{aligned}
R_{\rho\sigma\mu\nu} &= \frac{1}{2} (\partial_\mu \partial_\sigma g_{\nu\rho} - \partial_\mu \partial_\rho g_{\sigma\nu} - \partial_\nu \partial_\sigma g_{\mu\rho} + \partial_\nu \partial_\rho g_{\sigma\mu}) \\
&= \frac{1}{2} (\partial_\rho \partial_\nu g_{\sigma\mu} - \partial_\rho \partial_\mu g_{\nu\sigma} - \partial_\sigma \partial_\nu g_{\rho\mu} + \partial_\sigma \partial_\mu g_{\nu\rho}) \\
&= R_{\mu\nu\rho\sigma}
\end{aligned} \tag{5}$$

1.5 Cyclic Identity

This is unfortunately also called “first Bianchi identity”.

$$\begin{aligned}
R_{\rho\sigma\mu\nu} + R_{\rho\mu\nu\sigma} + R_{\rho\nu\sigma\mu} &= \frac{1}{2} (\partial_\mu \partial_\sigma g_{\rho\nu} - \partial_\mu \partial_\rho g_{\sigma\nu} - \partial_\nu \partial_\sigma g_{\mu\rho} + \partial_\nu \partial_\rho g_{\sigma\mu}) \\
&\quad + \frac{1}{2} (\partial_\nu \partial_\mu g_{\rho\sigma} - \partial_\nu \partial_\rho g_{\sigma\mu} - \partial_\sigma \partial_\mu g_{\nu\rho} + \partial_\rho \partial_\sigma g_{\nu\mu}) \\
&\quad + \frac{1}{2} (\partial_\nu \partial_\sigma g_{\rho\mu} - \partial_\rho \partial_\sigma g_{\mu\nu} - \partial_\mu \partial_\nu g_{\rho\sigma} + \partial_\rho \partial_\mu g_{\sigma\nu}) \\
&= 0
\end{aligned} \tag{6}$$

1.6 Bianchi Identity

Again, in local inertial coordinates

$$\begin{aligned}
\nabla_\lambda R_{\rho\sigma\mu\nu} + \nabla_\sigma R_{\lambda\rho\mu\nu} + \nabla_\rho R_{\sigma\lambda\mu\nu} &= \partial_\lambda R_{\rho\sigma\mu\nu} + \partial_\sigma R_{\lambda\rho\mu\nu} + \partial_\rho R_{\sigma\lambda\mu\nu} \\
&= \frac{1}{2} \partial_\lambda (\partial_\mu \partial_\sigma g_{\nu\rho} - \partial_\mu \partial_\rho g_{\sigma\nu} - \partial_\nu \partial_\sigma g_{\mu\rho} + \partial_\nu \partial_\rho g_{\sigma\mu}) \\
&\quad + \frac{1}{2} \partial_\sigma (\partial_\mu \partial_\rho g_{\nu\lambda} - \partial_\mu \partial_\lambda g_{\nu\rho} - \partial_\nu \partial_\rho g_{\lambda\mu} + \partial_\nu \partial_\lambda g_{\rho\mu}) \\
&\quad + \frac{1}{2} \partial_\rho (\partial_\mu \partial_\lambda g_{\nu\sigma} - \partial_\mu \partial_\sigma g_{\nu\lambda} - \partial_\nu \partial_\lambda g_{\mu\sigma} + \partial_\nu \partial_\sigma g_{\mu\lambda}) \\
&= \frac{1}{2} (\partial_\lambda \partial_\mu \partial_\sigma g_{\nu\rho} - \partial_\lambda \partial_\mu \partial_\rho g_{\sigma\nu} - \partial_\lambda \partial_\nu \partial_\sigma g_{\mu\rho} + \partial_\lambda \partial_\nu \partial_\rho g_{\sigma\mu}) \\
&\quad + \frac{1}{2} (\partial_\sigma \partial_\mu \partial_\rho g_{\nu\lambda} - \partial_\sigma \partial_\mu \partial_\lambda g_{\nu\rho} - \partial_\sigma \partial_\nu \partial_\rho g_{\lambda\mu} + \partial_\sigma \partial_\nu \partial_\lambda g_{\rho\mu}) \\
&\quad + \frac{1}{2} (\partial_\rho \partial_\mu \partial_\lambda g_{\nu\sigma} - \partial_\rho \partial_\mu \partial_\sigma g_{\nu\lambda} - \partial_\rho \partial_\nu \partial_\lambda g_{\mu\sigma} + \partial_\rho \partial_\nu \partial_\sigma g_{\mu\lambda}) \\
&= 0
\end{aligned} \tag{7}$$

1.7 Ricci Tensor Is Symmetric

Prove with the pairwise symmetry of Riemann (5) and the symmetry of the metric

$$R_{\sigma\nu} = g^{\rho\mu} R_{\rho\sigma\mu\nu} = g^{\mu\rho} R_{\mu\nu\rho\sigma} = R_{\nu\sigma} \tag{8}$$

1.8 Contractions Of Riemann

The metric is symmetric, so there are $\binom{4}{2} = 6$ pairs of contractions to consider.

$$g^{\rho\mu} R_{\rho\sigma\mu\nu} = R_{\sigma\nu} \tag{9}$$

$$g^{\rho\sigma} R_{\rho\sigma\mu\nu} = 0 \tag{10}$$

$$g^{\mu\nu} R_{\rho\sigma\mu\nu} = 0 \tag{11}$$

since Riemann is antisymmetric in its first two indices (3) and its last two indices (4), while the metric is symmetric, and contraction of symmetric indices with antisymmetric indices vanish.

$$g^{\sigma\nu} R_{\rho\sigma\mu\nu} = -g^{\sigma\nu} R_{\sigma\rho\mu\nu} = +g^{\sigma\nu} R_{\sigma\rho\nu\mu} = R_{\rho\mu} \tag{12}$$

$$g^{\rho\nu} R_{\rho\sigma\mu\nu} = -g^{\rho\nu} R_{\rho\sigma\nu\mu} = -R_{\sigma\mu} \tag{13}$$

$$g^{\sigma\mu}R_{\rho\sigma\mu\nu} = -g^{\sigma\mu}R_{\sigma\rho\mu\nu} = -R_{\rho\nu} \quad (14)$$

1.9 Contracted Bianchi Identity

We take Bianchi identity (7) and contract it twice with the inverse metric $g^{\rho\mu}g^{\sigma\nu}$. This specific contractions is chosen such that the first term will produce the Ricci scalar. We use metric compatibility $\nabla g = 0$ to insert the metric into the covariant derivative.

$$\begin{aligned} 0 &= g^{\rho\mu}g^{\sigma\nu}(\nabla_\lambda R_{\rho\sigma\mu\nu} + \nabla_\sigma R_{\lambda\rho\mu\nu} + \nabla_\rho R_{\sigma\lambda\mu\nu}) = \nabla_\lambda R - \nabla^\nu R_{\lambda\nu} - \nabla^\mu R_{\lambda\mu} \\ &= \nabla^\nu (g_{\lambda\nu}R) - 2\nabla^\nu R_{\lambda\nu} = \nabla^\nu (g_{\lambda\nu}R - 2R_{\lambda\nu}) \end{aligned} \quad (15)$$

Multiply by $-\frac{1}{2}$

$$\nabla^\nu \left(R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} \right) = 0 \quad (16)$$

2 The Hyperbolic Plane

2.1 The Poincare Half Plane

$$ds^2 = h^2(y)(dx^2 + dy^2) \quad (17)$$

The metric components are

$$g_{xx} = g_{yy} = h^2 \quad g_{xy} = g_{yx} = 0 \quad (18)$$

$$g^{xx} = g^{yy} = h^{-2} \quad g^{xy} = g^{yx} = 0 \quad (19)$$

The six independent Christoffel symbols are

$$\Gamma_{xx}^x = 0 \quad (20)$$

$$\Gamma_{yy}^x = 0 \quad (21)$$

$$\Gamma_{xy}^x = \frac{1}{2}g^{xx}\partial_y g_{xx} = \frac{h'}{h} \quad (22)$$

$$\Gamma_{xx}^y = -\frac{1}{2}g^{yy}\partial_y g_{xx} = -\frac{h'}{h} \quad (23)$$

$$\Gamma_{yy}^y = \frac{1}{2}g^{yy}\partial_y g_{yy} = \frac{h'}{h} \quad (24)$$

$$\Gamma_{xy}^y = 0 \quad (25)$$

Riemann tensor:

$$R^x_{xxy} = R^y_{yxy} = 0 \quad (26)$$

because the metric is diagonal.

$$\begin{aligned} R^x_{yxy} &= \partial_x \Gamma_{yy}^x - \partial_y \Gamma_{yx}^x + \Gamma_{xk}^x \Gamma_{yy}^k - \Gamma_{yk}^x \Gamma_{yx}^k \\ &= -\partial_y \Gamma_{yx}^x + \Gamma_{xy}^x \Gamma_{yy}^y - \Gamma_{yx}^x \Gamma_{yx}^x = -\partial_y \Gamma_{yx}^x + \Gamma_{xy}^x (\Gamma_{yy}^y - \Gamma_{yx}^x) \\ &= -\left(\frac{h'}{h}\right)' + \frac{h'}{h} \left(\frac{h'}{h} - \frac{h'}{h}\right) = -\left(\frac{h'}{h}\right)' \end{aligned} \quad (27)$$

$$\begin{aligned} R^y_{xyx} &= \partial_y \Gamma_{xx}^y - \partial_x \Gamma_{xy}^y + \Gamma_{yk}^y \Gamma_{xx}^k - \Gamma_{xk}^y \Gamma_{xy}^k \\ &= \partial_y \Gamma_{xx}^y + \Gamma_{yy}^y \Gamma_{xx}^y - \Gamma_{xx}^y \Gamma_{xy}^x = \partial_y \Gamma_{xx}^y + \Gamma_{xx}^y (\Gamma_{yy}^y - \Gamma_{xy}^x) \\ &= -\left(\frac{h'}{h}\right)' - \frac{h'}{h} \left(\frac{h'}{h} - \frac{h'}{h}\right) = -\left(\frac{h'}{h}\right)' \end{aligned} \quad (28)$$

Ricci tensor:

$$R_{xx} = R^k_{xkx} = R^y_{xyx} = -\left(\frac{h'}{h}\right)' \quad (29)$$

$$R_{yy} = R^k_{yky} = R^x_{yxy} = -\left(\frac{h'}{h}\right)' \quad (30)$$

$$R_{xy} = 0 \quad (31)$$

Ricci Scalar:

$$R = g^{ij} R_{ij} = g^{xx} R_{xx} + g^{yy} R_{yy} = -2h^{-2} \left(\frac{h'}{h}\right)' \quad (32)$$

The Ricci scalar of the hyperbolic plane is

$$R = -\frac{2}{a^2} \quad (33)$$

Therefore

$$h^{-2} \left(\frac{h'}{h}\right)' = \frac{1}{a^2} \quad (34)$$

Guess the form

$$h(y) = Ay^\alpha \quad (35)$$

$$h' = \alpha Ay^{\alpha-1} \quad (36)$$

then

$$\frac{h'}{h} = \frac{\alpha}{y} \quad (37)$$

plug in (34)

$$A^{-2}y^{-2\alpha} \left(-\frac{\alpha}{y^2} \right) = \frac{1}{a^2} \quad (38)$$

$$-\alpha A^{-2}y^{-2\alpha-2} = a^{-2} \quad (39)$$

By equating powers we have

$$\alpha = -1 \quad (40)$$

then

$$A = a \quad (41)$$

$$h(y) = \frac{a}{y} \quad (42)$$

$$ds^2 = \frac{a^2}{y^2} (dx^2 + dy^2) \quad (43)$$

Another way is to calculate only R_{xyxy} and obtain R_{yxyx} from symmetry.

A third short way in two dimensions is to obtain R directly from R^x_{yxy} .
 $R = \frac{2R^x_{yxy}}{g_{yy}} = -2h^{-2} \left(\frac{h'}{h} \right)'$.

2.2 The Poincare Disc

$$ds^2 = \frac{4a^4}{(a^2 - r^2)^2} (dr^2 + r^2 d\theta^2) \quad (44)$$

Denote

$$q(r) = \frac{2a^2}{a^2 - r^2} \quad (45)$$

The metric components are

$$g_{rr} = q^2 \quad g_{\theta\theta} = q^2 r^2 \quad g_{r\theta} = 0 \quad (46)$$

$$g^{rr} = q^{-2} \quad g^{\theta\theta} = q^{-2} r^{-2} \quad g^{r\theta} = 0 \quad (47)$$

The Christoffel symbols are

$$\Gamma_{rr}^r = \frac{1}{2}g^{rr}\partial_r g_{rr} = \frac{q'}{q} \quad (48)$$

$$\Gamma_{\theta\theta}^r = -\frac{1}{2}g^{rr}\partial_r g_{\theta\theta} = -r - r^2\frac{q'}{q} \quad (49)$$

$$\Gamma_{r\theta}^r = 0 \quad (50)$$

$$\Gamma_{rr}^\theta = 0 \quad (51)$$

$$\Gamma_{\theta\theta}^\theta = 0 \quad (52)$$

$$\Gamma_{\theta r}^\theta = \frac{1}{2}g^{\theta\theta}\partial_r g_{\theta\theta} = \frac{1}{r} + \frac{q'}{q} \quad (53)$$

Riemann tensor: This time we shall calculate directly only the component $R_{r\theta r\theta}$

$$\begin{aligned} R_{r\theta r\theta} &= g_{rk}R_{\theta r\theta}^k = g_{rr}R_{\theta r\theta}^r \\ &= g_{rr}(\partial_r\Gamma_{\theta\theta}^r - \partial_\theta\Gamma_{\theta r}^r + \Gamma_{rk}^r\Gamma_{\theta\theta}^k - \Gamma_{\theta k}^r\Gamma_{\theta r}^k) \\ &= g_{rr}[\partial_r\Gamma_{\theta\theta}^r + \Gamma_{rr}^r\Gamma_{\theta\theta}^r - \Gamma_{\theta\theta}^r\Gamma_{\theta r}^\theta] = g_{rr}[\partial_r\Gamma_{\theta\theta}^r + \Gamma_{\theta\theta}^r(\Gamma_{rr}^r - \Gamma_{\theta r}^\theta)] \\ &= q^2\left[\partial_r\left(-r - r^2\frac{q'}{q}\right) + \left(-r - r^2\frac{q'}{q}\right)\left(\frac{q'}{q} - \left(\frac{1}{r} + \frac{q'}{q}\right)\right)\right] \\ &= -q^2\left[r\frac{q'}{q} + r^2\left(\frac{q'}{q}\right)'\right] \end{aligned} \quad (54)$$

$$q' = \frac{4a^2r}{(a^2 - r^2)^2} \quad (55)$$

$$\frac{q'}{q} = \frac{2r}{a^2 - r^2} \quad (56)$$

$$\left(\frac{q'}{q}\right)' = \frac{2(a^2 - r^2) + 4r^2}{(a^2 - r^2)^2} = \frac{2(a^2 + r^2)}{(a^2 - r^2)^2} \quad (57)$$

$$\begin{aligned} R_{r\theta r\theta} &= -q^2\left[\frac{2r^2}{a^2 - r^2} + r^2\frac{2(a^2 + r^2)}{(a^2 - r^2)^2}\right] = -4q^2r^2\frac{a^2}{(a^2 - r^2)^2} \\ &= -r^2q^4\frac{1}{a^2} \end{aligned} \quad (58)$$

Ricci Tensor:

$$\begin{aligned}
R_{rr} &= g^{ij} R_{irjr} = g^{\theta\theta} R_{\theta r\theta r} = g^{\theta\theta} R_{r\theta r\theta} \\
&= q^{-2} r^{-2} \left[-r^2 q^4 \frac{1}{a^2} \right] = -\frac{q^2}{a^2}
\end{aligned} \tag{59}$$

$$\begin{aligned}
R_{\theta\theta} &= g^{ij} R_{i\theta j\theta} = g^{rr} R_{r\theta r\theta} \\
&= q^{-2} \left[-r^2 q^4 \frac{1}{a^2} \right] = -r^2 \frac{q^2}{a^2}
\end{aligned} \tag{60}$$

$$\begin{aligned}
R &= g^{ij} R_{ij} = g^{rr} R_{rr} + g^{\theta\theta} R_{\theta\theta} \\
&= q^{-2} \left(-\frac{q^2}{a^2} \right) + q^{-2} r^{-2} \left(-r^2 \frac{q^2}{a^2} \right) = -\frac{2}{a^2}
\end{aligned} \tag{61}$$

Yes, we got the Ricci scalar of the hyperbolic plane.

$$R = -\frac{2}{a^2}$$

Shortcut for two dimensions:

$$R = \frac{2R_{r\theta r\theta}}{g} = \frac{2(-r^2 q^4 \frac{1}{a^2})}{q^4 r^2} = -\frac{2}{a^2} \tag{62}$$

3 A Uniform Gravitational Field

$$ds^2 = -(1 + gx)^2 dt^2 + dx^2 + dy^2 + dz^2 \tag{63}$$

Calculate and find $R^\rho_{\sigma\mu\nu} = 0$.

Rewrite the metric as

$$ds^2 = -\left(\frac{1}{g} + x\right)^2 d(gt)^2 + d\left(\frac{1}{g} + x\right)^2 + dy^2 + dz^2 \tag{64}$$

$$y' = y \quad z' = z \tag{65}$$

Recognize flat metric in hyper-polar coordinates of radius $\left(\frac{1}{g} + x\right)$ and angle gt . The Cartesian coordinates (x', t') are

$$x' = \left(\frac{1}{g} + x\right) \cosh(gt) \tag{66}$$

$$t' = \left(\frac{1}{g} + x \right) \sinh(gt) \quad (67)$$

$$dx' = \cosh(gt) dx + (1 + gx) \sinh(gt) dt \quad (68)$$

$$dt' = \sinh(gt) dx + (1 + gx) \cosh(gt) dt \quad (69)$$

Leaving out $dy'^2 + dz'^2$, we have

$$\begin{aligned} ds^2 &= -dt'^2 + dx'^2 = -\sinh^2(gt) dx^2 - (1 + gx)^2 \cosh^2(gt) dt^2 - 2\sinh(gt) dx (1 + gx) \cosh(gt) dt \\ &\quad + \cosh^2(gt) dx^2 + (1 + gx)^2 \sinh^2(gt) dt^2 + 2\cosh(gt) dx (1 + gx) \sinh(gt) dt \\ &= -(1 + gx)^2 dt^2 + dx^2 \end{aligned} \quad (70)$$

4 Curvature Of The Schwarzschild Metric

4.1 Riemann Tensor

See at the end of Hartle's book.

4.2 Ricci Scalar

Schwarzschild metric is a solution to the vacuum Einstein equation, so $R_{\mu\nu} = 0$. Therefore the Ricci scalar is also zero $R = 0$.

4.3 Some Scalars

The scalars involving the Ricci tensor vanish.

$$R_{\mu\nu}R^{\mu\nu} = 0 \quad R_{\mu\nu\rho\sigma}R^{\mu\rho}R^{\nu\sigma} = 0 \quad (71)$$

The Kretschmann scalar is

$$R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} = \frac{48M^2}{r^6} \quad (72)$$

4.4 Divergence

The Kretschmann scalar diverges at $r = 0$, which shows that this is a real singularity, and not a coordinate singularity (since a scalar is the same in all coordinates).

5 Geodesic Deviation Equation

S^ρ satisfies the geodesic deviation equation

$$(\nabla_u \nabla_u S)^\rho = R^\rho_{\sigma\mu\nu} u^\sigma u^\mu S^\nu \quad (73)$$

Given a vector S'^ρ

$$S'^\rho = S^\rho + cu^\rho \quad (74)$$

then

$$S^\rho = S'^\rho - cu^\rho \quad (75)$$

Plug in (73)

$$(\nabla_u \nabla_u (S' - cu))^\rho = R^\rho_{\sigma\mu\nu} u^\sigma u^\mu (S'^\nu - cu^\nu) \quad (76)$$

$$(\nabla_u \nabla_u S')^\rho - c(\nabla_u \nabla_u u)^\rho = R^\rho_{\sigma\mu\nu} u^\sigma u^\mu S'^\nu - cR^\rho_{\sigma\mu\nu} u^\sigma u^\mu u^\nu \quad (77)$$

but u satisfies is the geodesic equation

$$\nabla_u u = 0 \quad (78)$$

and

$$R^\rho_{\sigma\mu\nu} u^\mu u^\nu = 0 \quad (79)$$

because $R^\rho_{\sigma\mu\nu}$ is antisymmetric while $u^\mu u^\nu$ is symmetric in $\mu\nu$.

Therefore S' also satisfies the geodesic deviation equation

$$(\nabla_u \nabla_u S')^\rho = R^\rho_{\sigma\mu\nu} u^\sigma u^\mu S'^\nu \quad (80)$$