Exercise 11 - Solution

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Gravity 1 2021-2022

Contents

1 Linearized Gravity

1.1 Inverse Metric

The Linearized metric is

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} \tag{1}$$

The alleged linearized inverse metric is

$$g^{\mu\nu} = \eta^{\mu\nu} - h^{\mu\nu} \tag{2}$$

where

$$h^{\mu\nu} \equiv \eta^{\rho\mu}\eta^{\sigma\nu}h_{\rho\sigma} \tag{3}$$

Multiply the metric (1) and its presumed inverse (2)

$$\begin{aligned} g_{\mu\sigma}g^{\sigma\nu} &= \left(\eta_{\mu\sigma} + h_{\mu\sigma}\right)\left(\eta^{\sigma\nu} - h^{\sigma\nu}\right) \\ &= \eta_{\mu\sigma}\eta^{\sigma\nu} - \eta_{\mu\sigma}h^{\sigma\nu} + h_{\mu\sigma}\eta^{\sigma\nu} - h_{\mu\sigma}h^{\sigma\nu} \\ &= \delta_\mu^\nu + O\left(h^2\right) \end{aligned} \tag{4}$$

To first order we got the unit matrix, so indeed (2) is the linearized inverse.

1.2 Linearized Christoffel And Curvature Tensors

Christoffel symbol

$$\begin{aligned} \left(\Gamma^\rho_{\mu\nu}\right)^{(1)} &= \frac{1}{2}\left(\eta^{\rho\sigma} - h^{\rho\sigma}\right)\left(\partial_\mu\left(\eta_{\nu\sigma} + h_{\nu\sigma}\right) + \partial_\nu\left(\eta_{\mu\sigma} + h_{\mu\sigma}\right) - \partial_\sigma\left(\eta_{\mu\nu} + h_{\mu\nu}\right)\right) \\ &= \frac{1}{2}\eta^{\rho\sigma}\left(\partial_\mu h_{\nu\sigma} + \partial_\nu h_{\mu\sigma} - \partial_\sigma h_{\mu\nu}\right) \end{aligned} \tag{5}$$

since $\partial_\mu\eta_{\nu\sigma} = 0$ and $h^{\rho\sigma}\partial_\mu h_{\nu\sigma} \sim O(2)$.

Riemann tensor

$$\begin{aligned} \left(R_{\rho\sigma\mu\nu}\right)^{(1)} &= \left(\eta_{\rho\lambda} + h_{\rho\lambda}\right)\left(\partial_\mu\left(\Gamma^\lambda_{\sigma\nu}\right)^{(1)} - \partial_\nu\left(\Gamma^\lambda_{\sigma\mu}\right)^{(1)} + \left(\Gamma^\lambda_{\mu\alpha}\right)^{(1)}\left(\Gamma^\alpha_{\sigma\nu}\right)^{(1)} - \left(\Gamma^\lambda_{\nu\alpha}\right)^{(1)}\left(\Gamma^\alpha_{\sigma\mu}\right)^{(1)}\right) \\ &= \eta_{\rho\lambda}\left(\partial_\mu\left(\Gamma^\lambda_{\sigma\nu}\right)^{(1)} - \partial_\nu\left(\Gamma^\lambda_{\sigma\mu}\right)^{(1)}\right) \\ &= \frac{1}{2}\left(\partial_\mu\partial_\sigma h_{\rho\nu} - \partial_\nu\partial_\sigma h_{\rho\mu} - \partial_\mu\partial_\rho h_{\sigma\nu} + \partial_\nu\partial_\rho h_{\sigma\mu}\right) \end{aligned} \tag{6}$$

Ricci tensor

$$
\begin{aligned}
(R_{\sigma\nu})^{(1)} = (\eta^{\rho\mu} - h^{\rho\mu}) (R_{\rho\sigma\mu\nu})^{(1)} = \eta^{\rho\mu} (R_{\rho\sigma\mu\nu})^{(1)} \\
= \frac{1}{2} (\partial_\mu \partial_\sigma h^\mu{}_\nu + \partial_\nu \partial_\mu h^\mu{}_\sigma - \partial_\nu \partial_\sigma h - \Box h_{\sigma\nu})
\end{aligned} \tag{7}
$$

Ricci scalar

$$
\begin{aligned}
R^{(1)} = \eta^{\sigma\nu} (R_{\sigma\nu})^{(1)} = \frac{1}{2} (\partial_\mu \partial_\nu h^{\mu\nu} + \partial_\nu \partial_\mu h^{\mu\nu} - \partial_\nu \partial^\nu h - \Box h^\nu{}_\nu) \\
= \partial_\mu \partial_\nu h^{\mu\nu} - \Box h
\end{aligned} \tag{8}
$$

Einstein tensor

$$
\begin{aligned}
(G_{\mu\nu})^{(1)} = (R_{\mu\nu})^{(1)} - \frac{1}{2} R^{(1)} \eta_{\mu\nu} \\
= \frac{1}{2} \left(\partial_\sigma \partial_\mu h^\sigma{}_\nu + \partial_\nu \partial_\sigma h^\sigma{}_\mu - \partial_\mu \partial_\nu h - \Box h_{\mu\nu} - \eta_{\mu\nu} \partial_\rho \partial_\sigma h^{\rho\sigma} + \eta_{\mu\nu} \Box h \right)
\end{aligned} \tag{9}
$$

1.3 Harmonic Gauge

Harmonic coordinates are a set of coordinates such that each of them is a harmonic **function**, i.e., satisfies the Laplace equation

$$
\Box x^\mu = 0 \tag{10}
$$

These are four equations, where the $\Box$ is acting on the functions x^1, x^2, x^3, x^4. The index μ is part of the name of the function under consideration.

First we use the definition $\Box = g^{\rho\sigma} \nabla_\rho \nabla_\sigma$, and that the covariant derivative reduces to the partial derivative when acting on a function

$$
\Box x^\mu = g^{\rho\sigma} \nabla_\rho \nabla_\sigma x^\mu = g^{\rho\sigma} \nabla_\rho (\partial_\sigma x^\mu) = g^{\rho\sigma} \nabla_\rho \delta^\mu_\sigma \tag{11}
$$

Now, the second covariant derivative is acting on an object with one lower index σ (again, the μ is only part of the name of the coordinate function x^μ)

$$
\Box x^\mu = g^{\rho\sigma} \nabla_\rho \delta^\mu_\sigma = g^{\rho\sigma} \partial_\rho \delta^\mu_\sigma - g^{\rho\sigma} \Gamma^\nu_{\rho\sigma} \delta^\mu_\nu = 0 - g^{\rho\sigma} \Gamma^\mu_{\rho\sigma} \tag{12}
$$

So the harmonic condition (10) is equivalent to

$$
\boxed{\Gamma^\mu_{\rho\sigma} g^{\rho\sigma} = 0} \tag{13}
$$

A second way to write the d'Alembertian is terms of the metric, $\Box = \frac{1}{\sqrt{|g|}}\partial_\nu\left(\sqrt{|g|}g^{\nu\sigma}\partial_\sigma\right)$

$$\begin{aligned}\Box x^\mu =& \frac{1}{\sqrt{|g|}}\partial_\nu\left(\sqrt{|g|}g^{\nu\sigma}\partial_\sigma x^\mu\right) = \frac{1}{\sqrt{|g|}}\partial_\nu\left(\sqrt{|g|}g^{\nu\sigma}\delta^\mu_\sigma\right)\\ =&\frac{1}{\sqrt{|g|}}\partial_\nu\left(\sqrt{|g|}g^{\nu\mu}\right)\end{aligned} \tag{14}$$

So the harmonic condition (10) is equivalent to

$$\partial_\nu\left(\sqrt{|g|}g^{\nu\mu}\right) = 0 \tag{15}$$

Let us derive the linearized harmonic condition from both expressions. From (13) we have

$$\begin{aligned}0 = \Gamma^\rho_{\mu\nu}g^{\mu\nu} = & \frac{1}{2}g^{\rho\sigma}\left(\partial_\mu g_{\nu\sigma} + \partial_\nu g_{\mu\sigma} - \partial_\sigma g_{\mu\nu}\right)g^{\mu\nu}\\ =&g^{\rho\sigma}g^{\mu\nu}\partial_\mu g_{\nu\sigma} - \frac{1}{2}g^{\rho\sigma}g^{\mu\nu}\partial_\sigma g_{\mu\nu}\\ \approx&\eta^{\sigma\rho}\eta^{\mu\nu}\partial_\mu h_{\nu\sigma} - \frac{1}{2}\eta^{\sigma\rho}\eta^{\mu\nu}\partial_\sigma h_{\mu\nu}\\ =&\partial_\mu h^{\mu\rho} - \frac{1}{2}\eta^{\sigma\rho}\partial_\sigma h\end{aligned} \tag{16}$$

$\Rightarrow$

$$\partial_\nu h^{\nu\mu} - \frac{1}{2}\eta^{\mu\nu}\partial_\nu h = 0 \tag{17}$$

where we noticed that the first two terms from Γ are identical because of the contractions.

From (15) we have

$$\begin{aligned}0 = \frac{1}{\sqrt{|g|}}\partial_\nu\left(\sqrt{|g|}g^{\nu\mu}\right) = & \left(\frac{1}{\sqrt{|g|}}\partial_\nu\sqrt{|g|}\right)g^{\nu\mu} + \partial_\nu g^{\nu\mu}\\ =&\left(\frac{1}{2}g^{\rho\sigma}\partial_\nu g_{\rho\sigma}\right)g^{\nu\mu} + \partial_\nu g^{\nu\mu}\\ \approx&\frac{1}{2}\eta^{\rho\sigma}\partial_\nu h_{\rho\sigma}\eta^{\nu\mu} - \partial_\nu h^{\nu\mu}\\ =&\frac{1}{2}\eta^{\nu\mu}\partial_\nu h - \partial_\nu h^{\nu\mu}\end{aligned} \tag{18}$$

$\Rightarrow$

$$\partial_\nu h^{\nu\mu} - \frac{1}{2}\eta^{\mu\nu}\partial_\nu h = 0 \tag{19}$$

where in the step to the second line we used the identity

$$\frac{1}{\sqrt{|g|}}\partial_\nu \sqrt{|g|} = \frac{1}{2}g^{\rho\sigma}\partial_\nu g_{\rho\sigma} \tag{20}$$

(see recitation 9 section 2.3.2), and the minus sign in the third line is from the inverse metric (2).

For new coordinates $x'^\mu = x^\mu + \xi^\mu$ to be harmonic as well, we require

$$0 = \Box x'^\mu = \Box\left(x^\mu + \xi^\mu\right) = \Box x^\mu + \Box\xi^\mu = 0 + \Box\xi^\mu \tag{21}$$

$$\Box\,\xi^\mu = 0 \tag{22}$$

The transformation functions ξ^μ need to be harmonic.

1.4 Linearized Vacuum Einstein Equation In Harmonic Gauge

Linearized Vacuum Einstein equation is

$$R^{(1)}_{\mu\nu} = 0 \tag{23}$$

From (7) it is

$$\frac{1}{2}\left(\partial_\rho\partial_\nu h^\rho{}_\mu + \partial_\mu\partial_\rho h^\rho{}_\nu - \partial_\mu\partial_\nu h - \Box h_{\mu\nu}\right) = 0 \tag{24}$$

Harmonic gauge (19) can be written with a lower index as

$$\partial_\nu h^\nu{}_\mu = \frac{1}{2}\partial_\mu h \tag{25}$$

Plug in (24)

$$\frac{1}{2}\left(\frac{1}{2}\partial_\nu\partial_\mu h + \frac{1}{2}\partial_\mu\partial_\nu h - \partial_\mu\partial_\nu h - \Box h_{\mu\nu}\right) = -\frac{1}{2}\,\Box\, h_{\mu\nu} = 0 \tag{26}$$

Therefore

$$\Box h_{\mu\nu} = 0$$

1.5 Linearized Einstein Equation With Sources In Harmonic Gauge

Trace-reversed perturbation

$$\overline{h}_{\mu\nu} = h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}h \tag{27}$$

$$\overline{h}^{\mu\nu} = h^{\mu\nu} - \frac{1}{2}\eta^{\mu\nu}h \tag{28}$$

Condition (19) is

$$\partial_\nu\left(h^{\mu\nu} - \frac{1}{2}\eta^{\mu\nu}h\right) = 0 \tag{29}$$

$$\boxed{\partial_\nu\overline{h}^{\mu\nu} = 0}$$

To write Einstein equation with sources we need the Linearized Einstein Tensor in harmonic gauge. We already found Ricci tensor in this gauge, namely $(R_{\mu\nu})^{(1)} = -\frac{1}{2}\Box h_{\mu\nu}$. Now we find the linearized Ricci scalar (8) in this gauge

$$R^{(1)} = \partial_\mu\partial_\nu h^{\mu\nu} - \Box h = \frac{1}{2}\partial_\mu\eta^{\mu\nu}\partial_\nu h - \Box h = -\frac{1}{2}\Box h \tag{30}$$

The linearized Einstein tensor in harmonic gauge is

$$\begin{aligned}(G_{\mu\nu})^{(1)} &= (R_{\mu\nu})^{(1)} - \frac{1}{2}\eta_{\mu\nu}R^{(1)} \\ &= -\frac{1}{2}\Box h_{\mu\nu} + \frac{1}{2}\frac{1}{2}\eta_{\mu\nu}\Box h \\ &= -\frac{1}{2}\Box\left(h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}h\right) \\ &= -\frac{1}{2}\Box\overline{h}_{\mu\nu}\end{aligned} \tag{31}$$

The Einstein equation $G_{\mu\nu} = 8\pi T_{\mu\nu}$ becomes

$$\boxed{\Box\overline{h}_{\mu\nu} = -16\pi T_{\mu\nu}} \tag{32}$$

2 Production Of Gravitational Waves By A Rotating Rod

We need to calculate the quadrupole moment of the rotating rod. One way is to do it in the rotating frame of the rod (x', y', z'). The center of mass is at the origin, and we align the rod along the x'-axes. It has a constant linear density $\rho = \frac{M}{L}$ between $-\frac{L}{2} < x' < \frac{L}{2}$. There is no mass in the y' and z' directions, so the only non-zero component of the quadrupole tensor is

$$I^{x'x'} = \int_{-\frac{L}{2}}^{\frac{L}{2}} \frac{M}{L} x'^2 dx' = \frac{1}{12} ML^2 \tag{33}$$

This is just the moment of inertia of a uniform rod about its center of mass.

In order to use the quadrupole formula we need to transform I^{ij} back to the inertial frame (x, y, z), where the rod is rotating with angular velocity Ω. Suppose it rotates counter-clockwise in the xy plane and that the frames coincide at $t = 0$. The inverse transformation is

$$x = cos\left(\Omega t\right) x' - sin\left(\Omega t\right) y' \tag{34}$$

$$y = sin\left(\Omega t\right) x' + cos\left(\Omega t\right) y' \tag{35}$$

$$z = z' \tag{36}$$

The tensor transformation is

$$I^{ij} = \frac{\partial x^i}{\partial x^{i'}} \frac{\partial x^j}{\partial x^{j'}} I^{i'j'} = \frac{\partial x^i}{\partial x'} \frac{\partial x^j}{\partial x'} I^{x'x'} \tag{37}$$

where the two summations reduce to a single term. Plug (34) and (35) into (37), we find

$$I^{xx} = \left(\frac{\partial x}{\partial x'}\right)^2 I^{x'x'} = \frac{1}{12} ML^2 cos^2\left(\Omega t\right) = \frac{1}{12} ML^2 \frac{1}{2}\left(1 + cos\left(2\Omega t\right)\right) \tag{38}$$

$$I^{xy} = \frac{\partial x}{\partial x'} \frac{\partial y}{\partial x'} I^{x'x'} = \frac{1}{12} ML^2 sin\left(\Omega t\right) cos\left(\Omega t\right) = \frac{1}{12} ML^2 \frac{1}{2} sin\left(2\Omega t\right) \tag{39}$$

$$I^{yy} = \left(\frac{\partial y}{\partial x'}\right)^2 I^{x'x'} = \frac{1}{12} ML^2 sin^2\left(\Omega t\right) = \frac{1}{12} ML^2 \frac{1}{2}\left(1 - cos\left(2\Omega t\right)\right) \tag{40}$$

$$I^{xz} = I^{yz} = I^{zz} = 0 \tag{41}$$

The metric perturbation of the gravitational wave is

$$\boxed{\overline{h}^{ij} = \frac{2}{r}\frac{d^2 I^{ij}}{dt^2} = \frac{ML^2\Omega^2}{3r}\begin{pmatrix} -cos\left(2\Omega\left(t-r\right)\right) & -sin\left(2\Omega\left(t-r\right)\right) & 0 \\ -sin\left(2\Omega\left(t-r\right)\right) & cos\left(2\Omega\left(t-r\right)\right) & 0 \\ 0 & 0 & 0 \end{pmatrix}} \tag{42}$$

Another way to calculate the quadrupole moment is to treat the rod as composed of infinitesimal point masses $dM = \frac{M}{L}dr$, where $0 < r < \frac{L}{2}$. Then we can use the result for binary stars, in the inertial frame. We found that for one binary

$$dI^{xx} = 2\left(dM\right)r^2 cos^2\left(\Omega t\right) = 2\frac{M}{L}drr^2 cos^2\left(\Omega t\right) \tag{43}$$

integrate binaries of radii $0 < r < \frac{L}{2}$

$$I^{xx} = \int_0^{\frac{L}{2}} 2\frac{M}{L}drr^2 cos^2\left(\Omega t\right) = \frac{1}{12}ML^2 cos^2\left(\Omega t\right) \tag{44}$$

and the same for I^{xy} and I^{yy}.

3 Energy Loss Of A Binary Star Due To Gravitational Radiation

3.1 Power Emitted

See recitation 11 section 3.

The Power emitted is

$$P = -\frac{2}{5}\left(\frac{M}{R}\right)^5 \tag{45}$$

3.2 Radial Velocity

Newton second law

$$\frac{M^2}{\left(2R\right)^2} = M\frac{v^2}{R} \tag{46}$$

The energy of the binary is

$$E = 2\left(\frac{1}{2}Mv^2\right) - \frac{M^2}{2R} = \frac{M^2}{4R} - \frac{M^2}{2R} = -\frac{M^2}{4R} \tag{47}$$

The power is

$$P = \frac{dE}{dt} = \frac{M^2}{4R^2}\frac{dR}{dt} \tag{48}$$

Compare to (45), yields the radial velocity,

$$\boxed{\frac{dR}{dt} = -\frac{8}{5}\left(\frac{M}{R}\right)^3} \tag{49}$$

3.3 Period Decrease

From (46) the orbital frequency is

$$\Omega = \left(\frac{M}{4R^3}\right)^{\frac{1}{2}} \tag{50}$$

The period is

$$T = \frac{2\pi}{\Omega} = \frac{4\pi R^{\frac{3}{2}}}{M^{\frac{1}{2}}} \tag{51}$$

Differentiate (51)

$$\frac{dT}{dt} = \frac{4\pi}{M^{\frac{1}{2}}}\frac{3}{2}R^{\frac{1}{2}}\frac{dR}{dt} \tag{52}$$

Plugin (49)

$$\frac{dT}{dt} = -\frac{4\pi}{M^{\frac{1}{2}}}\frac{3}{2}R^{\frac{1}{2}}\frac{8}{5}\left(\frac{M}{R}\right)^3 = -4\pi\frac{3}{2}\frac{8}{5}\frac{M^{\frac{5}{2}}}{R^{\frac{5}{2}}} \tag{53}$$

From (51), substitute $R^{-\frac{5}{2}} = \left(\frac{TM^{\frac{1}{2}}}{4\pi}\right)^{-\frac{5}{3}}$

$$\frac{dT}{dt} = -4\pi\frac{3}{2}\frac{8}{5}M^{\frac{5}{2}}\left(T^{-\frac{5}{3}}\right)\left(\frac{M^{\frac{1}{2}}}{4\pi}\right)^{-\frac{5}{3}} = -\pi\frac{48}{5}\left(\frac{4\pi M}{T}\right)^{\frac{5}{3}} \tag{54}$$

$$\boxed{\frac{dT}{dt} = -\frac{48}{5}\pi\left(\frac{4\pi M}{T}\right)^{\frac{5}{3}}} \tag{55}$$