Exercise 12 - Solution

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Gravity 1 2021-2022

Contents

1 Friedmann Equations

1.1 Alternative Form Of Einstein Equation

Einstein equation is

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = 8\pi T_{\mu\nu} \tag{1}$$

Take the trace

$$R - \frac{1}{2}R4 = 8\pi T \tag{2}$$

$$R = -8\pi T \tag{3}$$

Plug back into (1) yields

$$\boxed{R_{\mu\nu} = 8\pi\left(T_{\mu\nu} - \frac{1}{2}Tg_{\mu\nu}\right)} \tag{4}$$

Alternative form of Einstein equation

1.2 Friedmann From Einstein

Robertson Walker metric is

$$ds^2 = -dt^2 + a^2(t)\left[\frac{dr^2}{1-kr^2} + r^2 d\Omega^2\right] \tag{5}$$

The non zero components of the Ricci tensor are

$$R_{tt} = -3\frac{\ddot{a}}{a} \tag{6}$$

$$R_{rr} = \frac{a\ddot{a} + 2\dot{a}^2 + 2k}{1-kr^2} \tag{7}$$

$$R_{\theta\theta} = r^2\left(a\ddot{a} + 2\dot{a}^2 + 2k\right) \tag{8}$$

$$R_{\phi\phi} = r^2\left(a\ddot{a} + 2\dot{a}^2 + 2k\right)sin^2\theta \tag{9}$$

The energy-momentum tensor of a perfect fluid is

$$T_{\mu\nu} = \begin{pmatrix} \rho & 0 & 0 & 0 \\ 0 & & & \\ 0 & & pg_{ij} & \\ 0 & & & \end{pmatrix} \tag{10}$$

Its trace is

$$T = g^{\mu\nu}T_{\mu\nu} = g^{tt}T_{tt} + g^{ij}T_{ij} = -\rho + g^{ij}pg_{ij} = -\rho + 3p \tag{11}$$

Plug (6),(10),(11), and g_{tt} into (4), the tt Einstein equation is

$$\begin{aligned} -3\frac{\ddot{a}}{a} &= 8\pi\left(\rho - \frac{1}{2}\left(-\rho + 3p\right)\left(-1\right)\right) \\ &= 8\pi\frac{1}{2}\left(\rho + 3p\right) \end{aligned} \tag{12}$$

The second Friedmann equation is

$$\boxed{\frac{\ddot{a}}{a} = -\frac{4\pi}{3}\left(\rho + 3p\right)} \tag{13}$$

Second Friedmann equation

Write some spatial equation, for instance the rr equation (plug (7),(10),(11) and g_{rr} into (4))

$$\frac{a\ddot{a} + 2\dot{a}^2 + 2k}{1 - kr^2} = 8\pi\left(p\frac{a^2}{1 - kr^2} - \frac{1}{2}\left(-\rho + 3p\right)\frac{a^2}{1 - kr^2}\right) \tag{14}$$

$$\frac{\ddot{a}}{a} + 2\frac{\dot{a}^2}{a^2} + 2\frac{k}{a^2} = 4\pi\left(\rho - p\right) \tag{15}$$

Plugin $\frac{\ddot{a}}{a}$ into (13) yields the first Friedmann equation

$$\boxed{\frac{\dot{a}^2}{a^2} = \frac{8\pi}{3}\rho - \frac{k}{a^2}} \tag{16}$$

First Friedmann equation

Remark: the first Friedmann equation can also be obtained as the tt Einstein equation in the original form (1) (which includes R instead of T).

2 Model Universe

$$ds^2 = -dt^2 + \frac{t}{t_*}\left(dx^2 + dy^2 + dz^2\right) \tag{17}$$

a) The spatial metric is flat, $k = 0$.

b) We see that

$$a\left(t\right) = \left(\frac{t}{t_*}\right)^{\frac{1}{2}} \tag{18}$$

We saw in tutorial that this corresponds to a radiation dominated universe, not matter. For matter dominated universe $a\left(t\right) \propto t^{\frac{2}{3}}$. $a\left(t\right) \propto t^{\frac{2}{n}}$, so $n = 4$.

$n = 3(1+w)$. $w = \frac{1}{3}$. The equation of state is $p = \frac{1}{3}\rho$.

c) Friedmann equation with $k = 0$ is

$$\rho = \frac{3}{8\pi}\left(\frac{\dot{a}}{a}\right)^2 \tag{19}$$

From (18)

$$\frac{\dot{a}}{a} = \frac{1}{2t} \tag{20}$$

then

$$\boxed{\rho(t) = \frac{3}{32\pi t^2}} \tag{21}$$

d) The Hubble constant today is

$$H_0 = \frac{\dot{a}(t_0)}{a(t_0)} = \frac{1}{2t_0} \tag{22}$$

Given that $t_0 = 14 \times 10^9 yr$,

$$\boxed{H_0 = \left(28 \times 10^9 yr\right)^{-1} \approx 3.57 \times 10^{-11} yr^{-2}} \tag{23}$$

Current Hubble constant

3 De Sitter Metric In Static Coordinates

We consider flat 5-dimensional space in coordinates (u, w, x, y, z), with metric

$$ds_5^2 = -du^2 + dw^2 + dx^2 + dy^2 + dz^2 \tag{24}$$

De Sitter space is the maximally symmetric space that arises as the 4-dimensional hypersurface hyperboloid of constant radius α, (with positive squared radius, the hyperboloid outside the light cone)

$$-u^2 + w^2 + x^2 + y^2 + z^2 = \alpha^2 \tag{25}$$

To find the hyperboloid metric in static coordinates we use spherical coordinates (r, θ, ϕ) for the 3-dimensional space (x, y, z)

$$x = rsin\theta cos\phi \tag{26}$$

$$y = rsin\theta sin\phi \tag{27}$$

$$z = rcos\theta \tag{28}$$

The radial coordinate is

$$x^2 + y^2 + z^2 = r^2 \tag{29}$$

Plug (29) into (25)

$$-u^2 + w^2 + r^2 = \alpha^2 \tag{30}$$

Now for each r there is a hyperbola in the (u, w) space

$$-u^2 + w^2 = \alpha^2 - r^2 \tag{31}$$

The radius of each hyperbola is $\sqrt{\alpha^2 - r^2}$ and we use hyper-polar coordinates with hyperbolic angle $\frac{t}{\alpha}$

$$u = \sqrt{\alpha^2 - r^2} sinh \left(\frac{t}{\alpha} \right) \tag{32}$$

$$w = \sqrt{\alpha^2 - r^2} cosh \left(\frac{t}{\alpha} \right) \tag{33}$$

(26),(27),(28),(32),(33) define the coordinates (t, r, θ, ϕ).

Now we turn to the metric (24), and substitute flat 3d metric in spherical coordinates

$$dx^2 + dy^2 + dz^2 = dr^2 + r^2 d\Omega^2 \tag{34}$$

and flat 2d Minkowski metric in hyper-polar coordinates, of radius $R \equiv \sqrt{\alpha^2 - r^2}$ and hyperbolic angle $\beta \equiv \frac{t}{\alpha}$

$$-du^2 + dw^2 = (dR)^2 - R^2 (d\beta)^2 = \left(d \left(\sqrt{\alpha^2 - r^2} \right) \right)^2 - \left(\alpha^2 - r^2 \right) \left(d \left(\frac{t}{\alpha} \right) \right)^2 \tag{35}$$

(24) becomes

$$ds^2 = \left(d \left(\sqrt{\alpha^2 - r^2} \right) \right)^2 - \left(\alpha^2 - r^2 \right) \left(d \left(\frac{t}{\alpha} \right) \right)^2 + dr^2 + r^2 d\Omega^2 \tag{36}$$

We need the differential

$$d \left(\sqrt{\alpha^2 - r^2} \right) = \frac{-2rdr}{2\sqrt{\alpha^2 - r^2}} \tag{37}$$

thus the combination of the dr^2 terms in (36) is

$$\left(d\left(\sqrt{\alpha^2-r^2}\right)\right)^2+dr^2=\left(\frac{r^2}{\alpha^2-r^2}+1\right)dr^2=\frac{\alpha^2}{\alpha^2-r^2}dr^2=\frac{1}{1-\frac{r^2}{\alpha^2}}dr^2 \tag{38}$$

Plug (38) back to (36) yields

$$ds^2=-\left(1-\frac{r^2}{\alpha^2}\right)dt^2+\frac{1}{1-\frac{r^2}{\alpha^2}}dr^2+r^2d\Omega^2$$

$$ds^2=-f\left(r\right)dt^2+f^{-1}\left(r\right)dr^2+r^2d\Omega^2 \tag{39}$$

$$f\left(r\right)=1-\frac{r^2}{\alpha^2} \tag{40}$$

De Sitter metric in static coordinates

As $r\to 0$, $f\left(r\right)\to 1$ and (39) is approaching smoothly a flat Minkowski space in spherical coordinates. At $r=0$ it is flat space with the usual coordinate singularity of spherical coordinates at the origin.

At $r=\alpha$ there is a cosmological horizon surrounding the observer at $r=0$.

4 Schwarzschild - De Sitter Metric

A general static spherically symmeetric metric can be written in coordinates (t,r,θ,ϕ) as

$$ds^2=-e^{2\alpha(r)}dt^2+e^{2\beta(r)}dr^2+r^2d\Omega^2 \tag{41}$$

The non-zero components of the Ricci tensor are

$$R_{tt}=e^{2(\alpha-\beta)}\left[\partial_r^2\alpha+\left(\partial_r\alpha\right)^2-\partial_r\alpha\partial_r\beta+\frac{2}{r}\partial_r\alpha\right] \tag{42}$$

$$R_{tt}=\left[-\partial_r^2\alpha-\left(\partial_r\alpha\right)^2+\partial_r\alpha\partial_r\beta+\frac{2}{r}\partial_r\beta\right] \tag{43}$$

$$R_{\theta\theta}=e^{-2\beta}\left[r\left(\partial_r\beta-\partial_r\alpha\right)-1\right]+1 \tag{44}$$

$$R_{\phi\phi}=sin^2\theta R_{\theta\theta} \tag{45}$$

As in the derivation of the Schwarzschild metric, it is useful to look at the combination

$$e^{-2(\alpha-\beta)}R_{tt}+R_{rr}=\frac{2}{r}\partial_r\alpha+\frac{2}{r}\partial_r\beta \tag{46}$$

The vacuum Einstein equation with a cosmological constant Λ (in 4 dimensions. See recitation 12 section 4.1) is

$$R_{\mu\nu}=\Lambda g_{\mu\nu} \tag{47}$$

Plug (47) into (46)

$$e^{-2(\alpha-\beta)}\Lambda g_{tt}+\Lambda g_{rr}=\frac{2}{r}\partial_r\left(\alpha+\beta\right) \tag{48}$$

Substitute g_{tt} and g_{rr} from (41)

$$-e^{-2(\alpha-\beta)}\Lambda e^{2\alpha}+\Lambda e^{2\beta}=\frac{2}{r}\partial_r\left(\alpha+\beta\right) \tag{49}$$

$$0=\frac{2}{r}\partial_r\left(\alpha+\beta\right) \tag{50}$$

Thus

$$\boxed{\alpha=-\beta} \tag{51}$$

It is interesting that this is the same result as in the Schwarzschild derivation, only there the l.h.s was zero because each term R_{tt}, R_{rr} was zero separately.

Now write the $\theta\theta$ equation (47)

$$R_{\theta\theta}=\Lambda g_{\theta\theta} \tag{52}$$

Plug (44), (51) and $g_{\theta\theta}$ from (41)

$$e^{2\alpha}\left[-2r\partial_r\alpha-1\right]+1=\Lambda r^2 \tag{53}$$

$$e^{2\alpha}\left[2r\partial_r\alpha+1\right]=1-\Lambda r^2 \tag{54}$$

$$\partial_r\left(re^{2\alpha}\right)=1-\Lambda r^2 \tag{55}$$

Integrate and name the integration constant $-r_0$

$$re^{2\alpha}=r-\frac{\Lambda}{3}r^3-r_0 \tag{56}$$

$$e^{2\alpha} = 1 - \frac{r_0}{r} - \frac{\Lambda}{3} r^2 \tag{57}$$

With (57) and (51) we find (41) to be

Schwarzschild - de Sitter metric

$$ds^2 = -f(r)\, dt^2 + f^{-1}(r)\, dr^2 + r^2 d\Omega^2 \tag{58}$$

$$f(r) \equiv 1 - \frac{r_0}{r} - \frac{\Lambda}{3} r^2 \tag{59}$$

For $\Lambda = 0$ (58) becomes the Schwarzschild metric in Schwarzschild coordinates, so it is natural to identify $r_0 = 2M$. This is good since this is a static and spherically symmetric solution of the vacuum Einstein equation, so without the cosmological constant it must reduce to the Schwarzschild metric.

For $M = 0$ (58) reduces to the de-Sitter metric in static coordinates we found in (39), with $\alpha = \sqrt{\frac{3}{\Lambda}}$. This is a solution of the vacuum Einstein equation which is not only static and spherically symmetric, but maximally symmetric. With positive Λ it is de Sitter space. The black hole sitting at $r = 0$ (when $M \neq 0$) is destroying all the symmetries except the time translation and reflection, and the isotropicity of space around it.