Derivation of the Ricci tensor and scalar for the FRW metric

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The FRW metric is of the form

$$ds^2 = -dt^2 + a^2(t)\,\gamma_{ij}\left(x^k\right)dx^i dx^j \tag{1}$$

where γ_{ij} is the metric of the spatial 3-dim. maximally symmetric space. It is commonly written in (r, θ, ϕ) coordinates, where it is also diagonal

$$\gamma_{ij}dx^i dx^j = \frac{dr^2}{1-kr^2} + r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right) \tag{2}$$

We derive the Ricci tensor $R_{\mu\nu}$ in three steps: First we obtain a general result for a metric in the time-space splitting of the form (1). Second, we restrict the spatial space to be maximally symmetric (homogeneous and isotropic). Third, we choose to use coordinates where γ_{ij} is diagonal, and specifically let it be (2).

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1 Space-time splitting with a general spatial hypersurfaces

1.1 Christoffel symbols

Since $g_{tt} = -1$, $g_{ti} = 0$, and $g_{ij} = a^2(t)\gamma_{ij}(x^k)$, The non vanishing Christoffel symbols are

$$\Gamma^t_{ij} = -\frac{1}{2}g^{tt}\partial_t g_{ij} = a\dot{a}\gamma_{ij} \tag{3}$$

$$\Gamma^i_{jt} = \frac{1}{2}g^{ik}\partial_t g_{jk} = \frac{1}{2}a^{-2}\gamma^{ik}2a\dot{a}\gamma_{jk} = \frac{\dot{a}}{a}\delta^i_j \tag{4}$$

Useful to write the contraction

$$\Gamma^i_{it} = 3\frac{\dot{a}}{a} \tag{5}$$

$$\Gamma^i_{jk} = \frac{1}{2}\gamma^{il}\left(\partial_j\gamma_{kl} + \partial_k\gamma_{jl} - \partial_l\gamma_{jk}\right) = \left(\Gamma^i_{jk}\right)^{(3)} \tag{6}$$

where $\left(\Gamma^i_{jk}\right)^{(3)}$ means that it is the Christoffel symbols of the spatial 3-dim. space.

Now we start calculating the Ricci tensor components.

1.2 $R_{ti} = 0$

First we show that the R_{ti} components vanish.

$$R^\rho{}_{\sigma\mu\nu} = \partial_\mu\Gamma^\rho_{\sigma\nu} - \partial_\nu\Gamma^\rho_{\sigma\mu} + \Gamma^\rho_{\mu\lambda}\Gamma^\lambda_{\nu\sigma} - \Gamma^\rho_{\nu\lambda}\Gamma^\lambda_{\mu\sigma}$$

$$R_{ti} = R^\mu{}_{t\mu i} = R^t{}_{tti} + R^j{}_{tji} \tag{7}$$

$$R^t{}_{tti} = \partial_t\Gamma^t_{ti} - \partial_i\Gamma^t_{tt} + \Gamma^t_{t\lambda}\Gamma^\lambda_{it} - \Gamma^t_{i\lambda}\Gamma^\lambda_{tt} = 0 \tag{8}$$

$$\begin{aligned} R^j{}_{tji} &= \partial_j\Gamma^j_{ti} - \partial_i\Gamma^j_{tj} + \Gamma^j_{j\lambda}\Gamma^\lambda_{it} - \Gamma^j_{i\lambda}\Gamma^\lambda_{jt} \\ &= \partial_j\left(\frac{\dot{a}}{a}\delta^j_i\right) - \partial_i\left(\frac{\dot{a}}{a}\delta^j_j\right) + \Gamma^j_{jk}\Gamma^k_{it} - \Gamma^j_{ik}\Gamma^k_{jt} \\ &= \Gamma^j_{jk}\frac{\dot{a}}{a}\delta^k_i - \Gamma^j_{ik}\frac{\dot{a}}{a}\delta^k_j = \frac{\dot{a}}{a}\left(\Gamma^j_{ji} - \Gamma^j_{ij}\right) = 0 \end{aligned} \tag{9}$$

Therefore

$$R_{ti} = 0 \tag{10}$$

1.3 R_{tt}

R_{tt} is also independent of γ_{ij}.

$$\begin{aligned}
R_{tt} = R^{\mu}{}_{t\mu t} &= \cancelto{0}{R^{t}{}_{ttt}} + R^{i}{}_{tit} \\
&= \partial_i \Gamma^i_{tt} - \partial_t \Gamma^i_{ti} + \Gamma^i_{i\lambda}\Gamma^{\lambda}_{tt} - \Gamma^i_{t\lambda}\Gamma^{\lambda}_{it} = -\partial_t \Gamma^i_{ti} - \Gamma^i_{tj}\Gamma^j_{it} \\
&= -\partial_t \left(3\frac{\dot{a}}{a}\right) - \left(\frac{\dot{a}}{a}\delta^i_j\right)\left(\frac{\dot{a}}{a}\delta^j_i\right) = -3\frac{\ddot{a}}{a} + 3\frac{\dot{a}^2}{a^2} - 3\frac{\dot{a}^2}{a^2} = -3\frac{\ddot{a}}{a}
\end{aligned} \tag{11}$$

$$R_{tt} = -3\frac{\ddot{a}}{a}$$

1.4 R_{ij}

$$R_{ij} = R^{\mu}{}_{i\mu j} = R^{t}{}_{itj} + R^{k}{}_{ikj} \tag{12}$$

$$\begin{aligned}
R^{t}{}_{itj} &= \partial_t \Gamma^t_{ij} - \partial_j \Gamma^t_{it} + \Gamma^t_{t\lambda}\Gamma^{\lambda}_{ji} - \Gamma^t_{j\lambda}\Gamma^{\lambda}_{ti} \\
&= \partial_t \Gamma^t_{ij} - \Gamma^t_{jk}\Gamma^k_{ti} \\
&= \partial_t \left(a\dot{a}\gamma_{ij}\right) - a\dot{a}\gamma_{jk}\frac{\dot{a}}{a}\delta^k_i \\
&= \left(a\ddot{a} + \dot{a}^2\right)\gamma_{ij} - \gamma_{ji}\dot{a}^2 = a\ddot{a}\gamma_{ij}
\end{aligned} \tag{13}$$

Now the pretty part, with a nuance,

$$\begin{aligned}
R^{k}{}_{ikj} &= \partial_k \Gamma^k_{ij} - \partial_j \Gamma^k_{ik} + \Gamma^k_{k\lambda}\Gamma^{\lambda}_{ji} - \Gamma^k_{j\lambda}\Gamma^{\lambda}_{ki} \\
&= \partial_k \left(\Gamma^k_{ij}\right)^{(3)} - \partial_j \left(\Gamma^k_{ik}\right)^{(3)} + \left(\Gamma^k_{kl}\right)^{(3)}\left(\Gamma^l_{ji}\right)^{(3)} - \left(\Gamma^k_{jl}\right)^{(3)}\left(\Gamma^l_{ki}\right)^{(3)} + \Gamma^k_{kt}\Gamma^t_{ji} - \Gamma^k_{jt}\Gamma^t_{ki} \\
&= \left(R^{k}{}_{ikj}\right)^{(3)} + \Gamma^k_{kt}\Gamma^t_{ji} - \Gamma^k_{jt}\Gamma^t_{ki} \\
&= \left(R_{ij}\right)^{(3)} + \Gamma^k_{kt}\Gamma^t_{ji} - \Gamma^k_{jt}\Gamma^t_{ki} \\
&= \left(R_{ij}\right)^{(3)} + \left(3\frac{\dot{a}}{a}\right)\left(a\dot{a}\gamma_{ij}\right) - \left(\frac{\dot{a}}{a}\delta^k_j\right)\left(a\dot{a}\gamma_{ki}\right) \\
&= \left(R_{ij}\right)^{(3)} + 2\dot{a}^2\gamma_{ij}
\end{aligned} \tag{14}$$

Plug (13) and (14) into (12)

$$R_{ij} = \left(a\ddot{a} + 2\dot{a}^2\right)\gamma_{ij} + \left(R_{ij}\right)^{(3)} \tag{15}$$

Where $\left(R^i{}_{jkl}\right)^{(3)}$ is the Riemann tensor of the 3-dim. spatial space, and $\left(R^k{}_{ikj}\right)^{(3)} = \left(R_{ij}\right)^{(3)}$ is the Ricci tensor of the 3-dim. spatial space. Notice that $R^i{}_{jkl} \neq \left(R^i{}_{jkl}\right)^{(3)}$ and $R_{ij} \neq \left(R_{ij}\right)^{(3)}$ because the summations over t. (15) gives the 4-dimensional R_{ij} components in terms of the 3-dimensional spatial metric and spatial Ricci tensor.

We collect the results of the Ricci tensor for a general spatial metric γ_{ij}

Ricci tensor for co-moving coordinates in terms of the scale factor and a **general** spatial hypersurfaces

$$R_{ti} = 0 \tag{16}$$

$$R_{tt} = -3\frac{\ddot{a}}{a} \tag{17}$$

$$R_{ij} = \left(a\ddot{a} + 2\dot{a}^2\right)\gamma_{ij} + \left(R_{ij}\right)^{(3)} \tag{18}$$

1.5 R

Lets compute the Ricci scalar also

$$\begin{aligned}
R &= g^{\mu\nu}R_{\mu\nu} = g^{tt}R_{tt} + g^{ij}R_{ij} \\
&= 3\frac{\ddot{a}}{a} + a^{-2}\gamma^{ij}\left(\left(a\ddot{a} + 2\dot{a}^2\right)\gamma_{ij} + \left(R_{ij}\right)^{(3)}\right) \\
&= 3\frac{\ddot{a}}{a} + 3\left(\frac{\ddot{a}}{a} + 2\frac{\dot{a}^2}{a^2}\right) + \frac{R^{(3)}}{a^2} \\
&= 6\left[\frac{\ddot{a}}{a} + \left(\frac{\dot{a}}{a}\right)^2\right] + \frac{R^{(3)}}{a^2}
\end{aligned} \tag{19}$$

Ricci scalar for co-moving coordinates in terms of the scale factor and a **general** spatial hypersurfaces

$$R = 6\left[\frac{\ddot{a}}{a} + \left(\frac{\dot{a}}{a}\right)^2\right] + \frac{R^{(3)}}{a^2} \tag{20}$$

2 Maximally symmetric 3-dimensional spatial hypersurfaces

The FRW metric assumes a homogeneous and isotropic universe, i.e., a maximally symmetric 3-dimensional spatial hypersurface.

The Ricci scalar of a 3-dim. maximally symmetric space, with a unit radius of curvature, is

$$R^{(3)} = 6k \tag{21}$$

where $k = \{-1, 0, 1\}$ for open, flat and closed space. The Ricci tensor of a 3-dim. maximally symmetric space, with a unit radius of curvature, is

$$(R_{ij})^{(3)} = \frac{1}{3} R^{(3)} \gamma_{ij} = 2k\gamma_{ij} \tag{22}$$

Plug (22) into (15), and (21) into (20), we collect the results of the Ricci tensor and scalar for a maximally symmetric spatial metric γ_{ij}

$$R_{ti} = 0 \tag{23}$$

$$R_{tt} = -3\frac{\ddot{a}}{a} \tag{24}$$

$$R_{ij} = \left(a\ddot{a} + 2\dot{a}^2 + 2k\right)\gamma_{ij} \tag{25}$$

$$R = 6\left[\frac{\ddot{a}}{a} + \left(\frac{\dot{a}}{a}\right)^2 + \frac{k}{a^2}\right] \tag{26}$$

Ricci tensor and Ricci scalar of **FRW metric** in terms of the scale factor and the spatial metric (for **any** spatial coordinates)

3 Diagonal spatial metric and the common (r, θ, ϕ) coordinates

Since R_{ij} is proportional to the metric, for a diagonal spatial metric the only non-vanishing Ricci tensor components are only the diagonal ones.

$$R_{ii} = \left(a\ddot{a} + 2\dot{a}^2 + 2k\right)\gamma_{ii} \tag{27}$$

In the (r, θ, ϕ) coordinates (2), just plug in the metric components. It reads

$$R_{tt} = -3\frac{\ddot{a}}{a} \tag{28}$$

$$R_{rr} = \frac{\left(a\ddot{a} + 2\dot{a}^2 + 2k\right)}{1 - kr^2} \tag{29}$$

$$R_{\theta\theta} = \left(a\ddot{a} + 2\dot{a}^2 + 2k\right) r^2 \tag{30}$$

$$R_{\phi\phi} = \left(a\ddot{a} + 2\dot{a}^2 + 2k\right) r^2 \sin^2\theta \tag{31}$$

Ricci tensor of FRW metric in the common (r, θ, ϕ) coordinates