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**The dipolar interaction.** The direct dipolar interaction of two magnetic dipoles,  $\mathbf{m}_1$  and  $\mathbf{m}_2$ , separated by a distance  $\mathbf{r}$  reads

$$U = \frac{1}{r^3} \left( \mathbf{m}_1 \cdot \mathbf{m}_2 - 3(\mathbf{m}_1 \cdot \hat{\mathbf{r}})(\mathbf{m}_2 \cdot \hat{\mathbf{r}}) \right). \quad (4.1)$$

In order to estimate its magnitude, we take  $m_1 \simeq m_2 \simeq g\mu_B \simeq e\hbar/mc$ . Then, using  $a_0 = \hbar^2/me^2$  and  $\alpha = e^2/(\hbar c) = 1/137$ ,

$$\begin{aligned} U &\simeq \left( \frac{e\hbar}{mc} \right)^2 \frac{1}{r^3} = \left( \frac{a_0}{r} \right)^3 \frac{e^2}{a_0} \left( \frac{e\hbar}{mc} \right)^2 \frac{1}{e^2} \left( \frac{me^2}{\hbar^2} \right)^2 \\ &= \left( \frac{a_0}{r} \right)^3 \frac{e^2}{a_0} \left( \frac{1}{137} \right)^2 = \left( \frac{a_0}{r} \right)^3 \left( \frac{1}{137} \right)^2 \text{Rd}. \end{aligned} \quad (4.2)$$

Since 1 Ry=13.6 eV and  $r$  is about several Bohr radii, this energy is about  $10^{-4}$ eV (which amounts to a temperature of a few degrees), and is far too small to explain the typical magnetic energies.

\*\*\* **exercise:** What is the preferred direction of two identical magnetic dipoles interacting via the dipolar interaction? [Answer: The dipolar interaction is minimal when the two dipoles are parallel to one another and to the radius vector  $\mathbf{r}$ . If they are perpendicular to the radius vector, then they prefer to be anti-parallel.]

**The exchange energy.** Let us consider two electrons interacting via the Coulomb interaction alone (namely, we neglect spin-dependent interactions like the spin-orbit interaction, etc.). The Hamiltonian reads

$$\mathcal{H}(1, 2) = -\frac{\hbar^2}{2m} \frac{\partial^2}{\partial \mathbf{r}_1^2} - \frac{\hbar^2}{2m} \frac{\partial^2}{\partial \mathbf{r}_2^2} + V(\mathbf{r}_1) + V(\mathbf{r}_2) + \frac{e^2}{r_{12}}, \quad (4.3)$$

where  $V(\mathbf{r})$  is due to the ions, and  $r_{12} = |\mathbf{r}_1 - \mathbf{r}_2|$ . When there are two independent orbital states,  $\psi_a(\mathbf{r})$  and  $\psi_b(\mathbf{r})$ , the orbital part of the two-electron wave function can be

$$\begin{aligned}\Psi(1, 2) &= \frac{1}{\sqrt{2}} \left( \psi_a(1)\psi_b(2) + \psi_b(1)\psi_a(2) \right), \quad \text{singlet}, \\ \Psi(1, 2) &= \frac{1}{\sqrt{2}} \left( \psi_a(1)\psi_b(2) - \psi_b(1)\psi_a(2) \right), \quad \text{triplet}.\end{aligned}\tag{4.4}$$

Let us now calculate the total energy of the system. We have

$$\begin{aligned}\int \int d\mathbf{r}_1 d\mathbf{r}_2 \Psi^*(1, 2) \mathcal{H} \Psi(1, 2) &= \int \int d\mathbf{r}_1 d\mathbf{r}_2 |\psi_a(1)\psi_b(2)|^2 \mathcal{H} \\ &\pm \int \int d\mathbf{r}_1 d\mathbf{r}_2 \psi_a^*(1)\psi_b^*(2)\psi_a(2)\psi_b(1) \mathcal{H}.\end{aligned}\tag{4.5}$$

(We have used here the fact that the Hamiltonian is invariant under the change  $1 \leftrightarrow 2$ .)

The integral

$$J_{ab} = \int \int d\mathbf{r}_1 d\mathbf{r}_2 \psi_a^*(1)\psi_b^*(2)\psi_a(2)\psi_b(1) \mathcal{H},\tag{4.6}$$

is called the exchange integral. When  $\psi_a$  is orthogonal to  $\psi_b$ , only the Coulomb interaction contributes to this integral. Hence

$$J_{ab} = \int \int d\mathbf{r}_1 d\mathbf{r}_2 \psi_a^*(1)\psi_b^*(2)\psi_a(2)\psi_b(1) \frac{e^2}{r_{12}}.\tag{4.7}$$

Let us denote

$$\rho(\mathbf{r}) = \psi_a(\mathbf{r})\psi_b^*(\mathbf{r}).\tag{4.8}$$

Then the exchange integral can be written in the form

$$J_{ab} = \int \int d\mathbf{r}_1 d\mathbf{r}_2 \rho^*(\mathbf{r}_1) \frac{e^2}{r_{12}} \rho(\mathbf{r}_2).\tag{4.9}$$

Let us further denote

$$\phi(\mathbf{r}) = \int d\mathbf{r}' \frac{e^2}{|\mathbf{r} - \mathbf{r}'|} \rho(\mathbf{r}').\tag{4.10}$$

By its definition,  $\phi(\mathbf{r})$  satisfies the Poisson equation,

$$\Delta\phi(\mathbf{r}) = -4\pi e^2 \rho(\mathbf{r}).\tag{4.11}$$

Using this in Eq. (4.9), we have

$$J_{ab} = -\frac{1}{4\pi e^2} \int d\mathbf{r} (\Delta \phi^*(\mathbf{r})) \phi(\mathbf{r}) . \quad (4.12)$$

We can now use Green's theorem (in other words, integrate by parts). Since the surface contribution vanishes as  $|r| \rightarrow \infty$ , we obtain

$$J_{ab} = \frac{1}{4\pi e^2} \int d\mathbf{r} |\nabla \phi(\mathbf{r})|^2 > 0 . \quad (4.13)$$

It follows that the exchange energy is *positive*.

Returning to the expression for the energy, Eq. (4.5), we see that the spatial integration in the first term there can be written in the form

$$\begin{aligned} \int \int d\mathbf{r}_1 d\mathbf{r}_2 |\psi_a(1)\psi_b(2)|^2 \mathcal{H} &= \int d\mathbf{r} |\psi_a(\mathbf{r})|^2 \mathcal{H}_0 + \int d\mathbf{r} |\psi_b(\mathbf{r})|^2 \mathcal{H}_0 \\ &+ \int \int d\mathbf{r}_1 d\mathbf{r}_2 |\psi_a(1)\psi_b(2)|^2 \frac{e^2}{r_{12}} \equiv E_a + E_b + K_{ab} . \end{aligned} \quad (4.14)$$

The term  $K_{ab}$  is called the Coulomb integral. Here,  $\mathcal{H}_0 = -(\hbar^2/2m)\Delta + V(\mathbf{r})$  is the single electron part of the Hamiltonian.

In summary, we have found that the energy of the two electron system, which is described by a *spin-independent Hamiltonian* is given by  $E_a + E_b + K_{ab} + J_{ab}$  when the two electrons are in the symmetric spatial wave function, and by  $E_a + E_b + K_{ab} - J_{ab}$  when they are in the anti-symmetric one. This property of the two electron system can be written in terms of spin operators, in the form

$$\mathcal{H} = E_a + E_b + K_{ab} - \frac{1}{2} J_{ab} (1 + 4\mathbf{s}_a \cdot \mathbf{s}_b) . \quad (4.15)$$

In order to prove this statement, we note that

$$2\mathbf{s}_a \cdot \mathbf{s}_b = (\mathbf{s}_a + \mathbf{s}_b)^2 - s_a^2 - s_b^2 = s^2 - \frac{3}{2} . \quad (4.16)$$

In the singlet state,  $s = 0$ , and hence  $\mathbf{s}_a \cdot \mathbf{s}_b = -3/4$ ; in the triplet state,  $s = 1$  (and  $s^2 = s(s+1) = 2$ ), and therefore  $\mathbf{s}_a \cdot \mathbf{s}_b = 1/4$ .

The Heisenberg interaction is based on the above picture. It reads

$$\mathcal{H} = - \sum_{\langle ij \rangle} J_{ij} \mathbf{s}_i \cdot \mathbf{s}_j . \quad (4.17)$$

This Hamiltonian is the basis for most of the investigations in magnetism. When  $J$  is positive, the ground state of the system is *ferromagnetic*: the spins are aligned all in the same direction. When  $J$  is *negative*, the system has an *antiferromagnetic* order.

**Super-exchange.** Consider an atom with a single orbital (i.e., a single energy level and a single wave function). When this level is empty, the energy is zero; if one electron is on the level, then the energy is  $\varepsilon$ ; and if two electrons occupy the atom, then the energy is  $2\varepsilon + U$ , where  $U$  represents the Coulomb repulsion. We denote the single-electron wave function by  $|\mathbf{R}\rangle$ .

Now consider a molecule made of two such atoms. We denote the wave function of the electron when it is on the second atom by  $|\mathbf{R}'\rangle$ , and allow for *quantum mechanical tunneling* processes between the two atoms, such that the *single-electron* Hamiltonian, denoted  $h$ , has the matrix elements

$$\langle \mathbf{R} | h | \mathbf{R}' \rangle = \langle \mathbf{R}' | h | \mathbf{R} \rangle = -t , \quad (4.18)$$

in addition to

$$\langle \mathbf{R} | h | \mathbf{R} \rangle = \langle \mathbf{R}' | h | \mathbf{R}' \rangle = \varepsilon . \quad (4.19)$$

**\*\*\* exercise:** Find the eigen energies and the eigen functions of the molecule when the electrons are spinless.

When the two electrons are in the singlet state, their spatial (symmetric) wave functions are

$$\Phi_0 = \frac{1}{\sqrt{2}} \left( |\mathbf{R}\rangle |\mathbf{R}'\rangle + |\mathbf{R}'\rangle |\mathbf{R}\rangle \right) , \quad \Phi_1 = |\mathbf{R}\rangle |\mathbf{R}\rangle , \quad \Phi_2 = |\mathbf{R}'\rangle |\mathbf{R}'\rangle . \quad (4.20)$$

The matrix of the Hamiltonian in the singlet subspace is (assuming that  $\langle \mathbf{R} | \mathbf{R}' \rangle = 0$ ,  $\langle \mathbf{R} | \mathbf{R} \rangle = \langle \mathbf{R}' | \mathbf{R}' \rangle = 1$ )

$$\begin{bmatrix} H_{00} & H_{01} & H_{02} \\ H_{10} & H_{11} & H_{12} \\ H_{20} & H_{21} & H_{22} \end{bmatrix} = \begin{bmatrix} 2\varepsilon & -\sqrt{2}t & -\sqrt{2}t \\ -\sqrt{2}t & 2\varepsilon + U & 0 \\ -\sqrt{2}t & 0 & 2\varepsilon + U \end{bmatrix} . \quad (4.21)$$

This Hamiltonian matrix has the eigen energies

$$2\varepsilon + U , \quad 2\varepsilon + \frac{U}{2} \pm \sqrt{\frac{U^2}{4} + 4t^2} , \quad (4.22)$$

and therefore the ground state of the singlet state has the energy

$$E_g^{\text{sing}} = 2\varepsilon + \frac{U}{2} - \sqrt{\frac{U^2}{4} + 4t^2} \simeq 2\varepsilon - \frac{4t^2}{U}, \quad \text{for } t \ll U. \quad (4.23)$$

Namely, the electrostatic energy *reduces* the total energy of the two electrons. However, this forces them to have opposite spins. In other words, the electrostatic energy favors the *antiferromagnetic* state.

\*\*\* **exercise:** Derive explicitly Eqs. (4.21) and (4.22). Find the exact ground state of the singlet. Plot the probability to find the two electrons on the same atom in this state as function of  $U/t$  and explain the result.

**Curie-Weiss law and ferromagnetism.** Let us first re-visit the derivation leading to the Curie law. When we suppose that each atom behaves like a small magnet of moment  $\boldsymbol{\mu}$ , then in the magnetic field  $\mathbf{H}$  it acquires the energy  $-\boldsymbol{\mu} \cdot \mathbf{H}$ . We further assume that each atom is independent of its neighbors and can rotate freely under the effect of the temperature. Since the atom is localized, it satisfies the Boltzmann statistics, and therefore its average moment is given by

$$\langle \boldsymbol{\mu} \rangle = \frac{\int d\Omega \boldsymbol{\mu} e^{\beta \boldsymbol{\mu} \cdot \mathbf{H}}}{\int d\Omega e^{\beta \boldsymbol{\mu} \cdot \mathbf{H}}}, \quad (4.24)$$

where  $d\Omega$  is the element of the solid angle for rotation. The total magnetization of a system of  $N$  atoms will be  $N\langle \boldsymbol{\mu} \rangle$ , and its magnetic susceptibility will be in general a tensor (the derivative of  $\mu_i$  with respect to  $H_j$ ). Confining ourselves to cubic symmetry, we find

$$\chi = N \left\langle \frac{\partial \boldsymbol{\mu}}{\partial \mathbf{H}} \right\rangle \simeq \frac{N}{k_B T} \langle \boldsymbol{\mu} \cdot \boldsymbol{\mu} \rangle = \frac{1}{3} \frac{N \langle \mu^2 \rangle}{k_B T}, \quad \text{for small enough } H. \quad (4.25)$$

This is the Curie law.

\*\*\* **exercise:** Find the temperature dependence of the total magnetization of the system by using Eq. (4.24) in the limit of small magnetic fields, and compare with the previous results.

In many cases the interaction between neighboring atoms *cannot* be ignored. To account for this effect approximately, one may introduce an *internal* magnetic field exerted on each atom by its neighbors. This field is called the Weiss field (or sometimes molecular field). It is plausible to assume that the internal field is proportional to the average of the magnetic

moment, and that the proportionality constant reflects the strength of the inter-atom interactions. In other words,  $\lambda$  is related to the exchange coupling  $J$ , see Eq. (4.17). Hence, the internal magnetic field is

$$\mathbf{H}_I = \lambda N \langle \boldsymbol{\mu} \rangle . \quad (4.26)$$

The total field acting on each atom is now  $\mathbf{H} + \mathbf{H}_I$ , and therefore we find

$$\begin{aligned} N \langle \boldsymbol{\mu} \rangle &= N \frac{\int d\Omega \boldsymbol{\mu} e^{\beta \boldsymbol{\mu} \cdot (\mathbf{H} + \mathbf{H}_I)}}{\int d\Omega e^{\beta \boldsymbol{\mu} \cdot (\mathbf{H} + \mathbf{H}_I)}} \simeq \frac{N}{k_B T} \langle \boldsymbol{\mu} \boldsymbol{\mu} \cdot (\mathbf{H} + \mathbf{H}_I) \rangle \\ &\simeq \frac{N}{k_B T} \frac{\langle \mu^2 \rangle}{3} (\mathbf{H} + \lambda N \langle \boldsymbol{\mu} \rangle) . \end{aligned} \quad (4.27)$$

Let us define an effective temperature,  $\Theta$ ,

$$\Theta \equiv \frac{\lambda N \langle \mu^2 \rangle}{3 k_B} . \quad (4.28)$$

Then the susceptibility is

$$\chi = N \left\langle \frac{\partial \boldsymbol{\mu}}{\partial H} \right\rangle = \frac{N \langle \mu^2 \rangle}{3 k_B (T - \Theta)} . \quad (4.29)$$

This is the Curie-Weiss law for the magnetic susceptibility of a ferromagnet. In particular, the susceptibility diverges as the temperature approaches the ferromagnetic critical temperature, (the Curie temperature), which in our approximation is  $\Theta$ .

In order to show that  $\Theta$  is indeed the critical temperature of our model, we need to return to the full calculation of the magnetic moment. There we have found that the total magnetization, i.e.,  $N \langle \boldsymbol{\mu} \rangle$ , is given by the Brillouin function,  $B_J(x) = \frac{2J+1}{2J} \coth\left(\frac{(2J+1)x}{2J}\right) - \frac{1}{2J} \coth\left(\frac{x}{2J}\right)$ , with  $x = H/k_B T$ . For the case  $J = 1/2$  the Brillouin function reduces to  $\tanh$ , and hence

$$N \langle \boldsymbol{\mu} \rangle = N \tanh\left(\frac{H + H_I}{k_B T}\right) = N \tanh\left(\frac{H + \lambda N \langle \boldsymbol{\mu} \rangle}{k_B T}\right) . \quad (4.30)$$

Let us now consider this equation when the applied magnetic field  $H$  is zero. Obviously,  $\langle \boldsymbol{\mu} \rangle = 0$  is a solution for this equation. However, there might be another solution: since  $\tanh x \simeq x - x^3/3$  for small  $x$ , we see that as long as the temperature is less than  $\lambda N/k_B$  (which is up to a numerical constant equal to  $\Theta$  above), we can have a *nonzero* solution for the magnetization. In other words, the system sustains a spontaneous magnetic order below some critical temperature.

**Antiferromagnetism.** We may view an antiferromagnet as consisting of two sublattices. At zero temperature, the average magnetizations of the two sublattices are anti parallel to one another.

At finite temperatures, the internal fields acting on the two sublattices are different. The internal field acting on an atom belonging to the ‘-’ sublattice is

$$\mathbf{H}^- = \lambda N \boldsymbol{\mu}_+ , \quad (4.31)$$

while the internal field on an atom of the ‘+’ sublattice is

$$\mathbf{H}^+ = \lambda N \boldsymbol{\mu}_- , \quad (4.32)$$

where now  $\lambda$  is *negative*.

At sufficiently high temperatures, we can try to perform the same calculation as the one in Eqs. (4.27), (4.28), and (4.29). This will give us (note that  $N$  is the number of atoms in each sub lattice)

$$\begin{aligned} N \langle \boldsymbol{\mu}_+ \rangle &\simeq \frac{N}{k_B T} \frac{\langle \mu^2 \rangle}{3} (\mathbf{H} + \lambda N \langle \boldsymbol{\mu}_- \rangle) , \\ N \langle \boldsymbol{\mu}_- \rangle &\simeq \frac{N}{k_B T} \frac{\langle \mu^2 \rangle}{3} (\mathbf{H} + \lambda N \langle \boldsymbol{\mu}_+ \rangle) . \end{aligned} \quad (4.33)$$

Adding the two equations, we find that we have exactly the same result as before. However, in the present case,  $\lambda$  is negative, and therefore it is better to define

$$\Theta_N \equiv - \frac{\lambda N \langle \mu^2 \rangle}{3 k_B} , \quad (4.34)$$

to obtain

$$\chi = N \left\langle \frac{\partial \mu}{\partial H} \right\rangle = \frac{N \langle \mu^2 \rangle}{3 k_B (T + \Theta_N)} . \quad (4.35)$$

The paramagnetic susceptibility of an antiferromagnet is smaller than the one of a ferromagnet. The temperature  $\Theta_N$  is called the Néel temperature.

**Spin waves.** Let us consider the Heisenberg Hamiltonian for a *ferromagnet*,

$$\mathcal{H} = - \sum_{\langle \ell \ell' \rangle} J_{\ell \ell'} \mathbf{S}_\ell \cdot \mathbf{S}_{\ell'} . \quad (4.36)$$

We now assume that in the ground state of our system, all spins are aligned: they have their maximal possible value of  $S^z$ .

$$S_\ell^z |S\rangle_\ell = S |S\rangle_\ell . \quad (4.37)$$

The ground state itself is the product,

$$|0\rangle = |S\rangle_1 |S\rangle_2 \dots |S\rangle_N . \quad (4.38)$$

We now introduce the spin-deviation operators,  $S^\pm$ , (dropping the subscript  $\ell$  for brevity)

$$S^\pm = S^x \pm iS^y . \quad (4.39)$$

We can show that the application of these operators on the eigen states of  $S^z$  gives another eigen state of  $S^z$ ,

$$\begin{aligned} S^z (S^+ |S\rangle) &= S^z (S^x + iS^y) |S\rangle = (S^x S^z + iS^y + i[S^y S^z - iS^x]) |S\rangle \\ &= S^+ S^z |S\rangle + S^+ |S\rangle = (S + 1) (S^+ |S\rangle) . \end{aligned} \quad (4.40)$$

In a similar fashion,

$$S^z (S^- |S\rangle) = (S - 1) (S^- |S\rangle) . \quad (4.41)$$

Using the spin-deviation operators, we re-write the Heisenberg Hamiltonian in the form

$$\mathcal{H} = - \sum_{\langle \ell \ell' \rangle} J_{\ell \ell'} \left( S_\ell^z S_{\ell'}^z + \frac{1}{2} [S_\ell^+ S_{\ell'}^- + S_\ell^- S_{\ell'}^+] \right) . \quad (4.42)$$

As a result we see that

$$\mathcal{H}|0\rangle = - \sum_{\langle \ell \ell' \rangle} J_{\ell \ell'} S^2 |0\rangle , \quad (4.43)$$

namely, only the product of  $S^z$  in Eq. (4.42) contributes to the ground state energy.

What are the excitations of this system? One may think that an excited state is a state in which one spin deviates from being maximally aligned. However, this is not an eigen state, because the operation of  $S_\ell^- S_{\ell'}^+$  will shift the deviation to the nearest neighbor. In order to specify the excited states, we use the following description. Let us denote by  $|n\rangle$  a state which has  $n$  deviations ( $n < 2S$ ). Operating on this state with  $S^+$  will reduce  $n$  by 1, and



operating with  $S^-$  will increase it by 1. We see that in some sense  $S^+$  plays the role of an annihilation operator, denoted  $a$ , and  $S^-$  plays the role of a creation operator, denoted  $a^\dagger$ . In fact, we know that

$$[S^+, S^-] = i[S^y, S^x] - i[S^x, S^y] = 2S^z. \quad (4.44)$$

So, if we define

$$a = \frac{S^+}{\sqrt{2S^z}}, \quad a^\dagger = \frac{S^-}{\sqrt{2S^z}}, \quad (4.45)$$

we find that the creation and annihilation operators obey the usual commutation law,

$$[a, a^\dagger] = 1. \quad (4.46)$$

The definition (4.45) is not really proper, since  $S^z$  is also an operator. However, for large enough values of  $S$ , we can replace  $S^z$  by its average,  $S$ . This means that we may consider only small deviations from the ground state (and therefore this theory is not valid at all for spins 1/2).

Now the spin deviation states are eigen states of  $a^\dagger a$ , and we have

$$a^\dagger a |n\rangle = n |n\rangle, \quad a |n\rangle = \sqrt{n} |n-1\rangle, \quad a^\dagger |n\rangle = \sqrt{n+1} |n+1\rangle, \quad (4.47)$$

and

$$S^z = S - a^\dagger a. \quad (4.48)$$

Putting all this back into the Hamiltonian, Eq. (4.42), we find

$$\begin{aligned} \mathcal{H} &\simeq - \sum_{\langle \ell \ell' \rangle} J_{\ell \ell'} \left( [S - a_\ell^\dagger a_\ell] [S - a_{\ell'}^\dagger a_{\ell'}] + S [a_\ell^\dagger a_{\ell'} + a_\ell a_{\ell'}^\dagger] \right) \\ &\simeq - \sum_{\langle \ell \ell' \rangle} J_{\ell \ell'} \left( S^2 + S [a_\ell^\dagger a_{\ell'} + a_\ell a_{\ell'}^\dagger - a_\ell^\dagger a_\ell - a_{\ell'}^\dagger a_{\ell'}] \right). \end{aligned} \quad (4.49)$$

The first (non operator) term here is just the ground state, so we drop it when considering the excitations.

The Hamiltonian now resembles the one of lattice vibrations. As in that case, it is expedient to introduce the Fourier transforms

$$a_\ell = \sqrt{\frac{1}{N}} \sum_{\mathbf{q}} a_{\mathbf{q}} e^{i\mathbf{q} \cdot \mathbf{R}_\ell}, \quad a_\ell^\dagger = \sqrt{\frac{1}{N}} \sum_{\mathbf{q}} a_{\mathbf{q}}^\dagger e^{-i\mathbf{q} \cdot \mathbf{R}_\ell}. \quad (4.50)$$

( $\mathbf{R}_\ell$  is the radius-vector to site  $\ell$  of the lattice.) Then, taking into account that in Eq. (4.49)  $\ell$  and  $\ell'$  are nearest neighbors and that  $J_{\ell\ell'}$  depends only on  $|\mathbf{R}_\ell - \mathbf{R}_{\ell'}|$  we find

$$\mathcal{H} = \sum_{\mathbf{q}} \left( \sum_{\mathbf{R}_{nn}} 2SJ(1 - e^{-i\mathbf{q}\cdot\mathbf{R}_{nn}}) \right) a_{\mathbf{q}}^\dagger a_{\mathbf{q}} , \quad (4.51)$$

in which  $\mathbf{R}_{nn}$  is the radius-vector to the nearest neighbors.

Since the form (4.51) is exactly the same as encountered in the theory of lattice vibrations, we can simply use all the know results for this problem as well. Noting that the dispersion of the spin wave excitations,

$$\omega_q = \sum_{\mathbf{R}_{nn}} 2SJ(1 - e^{-i\mathbf{q}\cdot\mathbf{R}_{nn}}) \simeq 2SJq^2 a^2 , \quad (4.52)$$

we can find the average occupation number of each mode,

$$\langle a_{\mathbf{q}}^\dagger a_{\mathbf{q}} \rangle = \frac{1}{e^{\beta\omega_q/k_B T} - 1} , \quad (4.53)$$

is given by the Bose distribution. It follows that the temperature dependence of the magnetization,  $M(T)$ , is given by  $NS - \sum_{\mathbf{q}} \langle a_{\mathbf{q}}^\dagger a_{\mathbf{q}} \rangle$ .

\*\*\* **exercise:** Find an explicit form for the temperature dependence of the magnetization using the spin wave theory. Find the specific heat of the spin wave excitations.