SOLID-STATE PHYSICS II 2008

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The Meissner effect. An ideal superconductor permits no magnetic field in its interior. In other words, a superconductor is a perfect diamagnet. Not only a magnetic field is excluded from entering a superconductor, (which might have been explained by perfect conductivity), but a field in an originally normal (i.e., not superconductive) sample is expelled as the sample is cooled below the superconductive transition temperature, T_c. This fact *cannot* be explained by perfect conductivity. This is the Meissner effect.

On the other hand, the Meissner effect implies that superconductivity can be destroyed once the magnetic field exceeds a certain value, the critical magnetic field, H_c. Let us denote the free energy density of a system when it is above the superconducting transition temperature by f_n. As the system is cooled below T_c (in the absence of a magnetic field) it goes into the superconductive state, since in that state its free energy is lower. Let us denote that free energy density by f_s. Then the critical magnetic field is given by

$$\frac{H_c^2(T)}{8\pi} = f_n(T) - f_s(T) \ . \tag{5.1}$$

(Remember that the free energy of a magnetic field is given by the volume integral of $H^2/8\pi$.) Equation (5.1) implies that the critical magnetic field depends on the temperature, and it vanishes at $T = T_c$. In fact,

$$H_c(T) \simeq H_c(0)(1 - (T/T_c)^2) \ . \tag{5.2}$$

The Meissner effect is described by the (phenomenological) London equations. Let us consider the motion of the electrons in a perfect conductor, i.e., when they are accelerated by an electric field without any dissipation. Let us also assume that the number density of such 'dissipation-less' electrons is n_s, and their common velocity is $\mathbf{v}_s$. We can then write

$$m\frac{d\mathbf{v}_s}{dt} = -e\mathbf{E} \ , \tag{5.3}$$

where $\mathbf{E}$ is the electric field, and

$$\frac{d\mathbf{J}_s}{dt} = \frac{n_s e^2}{m}\mathbf{E} \ , \quad \mathbf{J}_s = -e\mathbf{v}_s n_s \ . \tag{5.4}$$

On the other hand, the Faraday's law,

$$\boldsymbol{\nabla} \times \mathbf{E} = -\frac{1}{c}\frac{\partial \mathbf{H}}{\partial t} \ , \tag{5.5}$$

gives

$$\frac{\partial}{\partial t}\left(\boldsymbol{\nabla} \times \mathbf{J}_s + \frac{n_s e^2}{mc}\mathbf{H}\right) = 0 \ . \tag{5.6}$$

The London equation states that in a *superconducting* system, not only the time derivative above vanishes, but that

$$\boldsymbol{\nabla} \times \mathbf{J}_s + \frac{n_s e^2}{mc}\mathbf{H} = 0 \ , \quad \text{London equation} \ . \tag{5.7}$$

Since the Maxwell equation gives us that

$$\boldsymbol{\nabla} \times \mathbf{H} = \frac{4\pi}{c}\mathbf{J}_s \ , \tag{5.8}$$

we arrive at the result that in a superconducting system

$$\nabla^2 \mathbf{H} = \frac{1}{\lambda^2}\mathbf{H} \ , \quad \text{or} \quad \nabla^2 \mathbf{J}_s = \frac{1}{\lambda^2}\mathbf{J}_s \ , \tag{5.9}$$

where the *penetration depth* λ is given by

$$\lambda^2 = \frac{mc^2}{4\pi n_s e^2} \ . \tag{5.10}$$

We note that the penetration depth *diverges* as the number density of the dissipation-less electrons (i.e., superconducting electrons), n_s, tends to zero. In other words, the penetration depth diverges as $T \to T_c$.

Let us consider a superconducting slab of finite thickness d, placed in a parallel magnetic field, H_a. The slab is perpendicular to the x direction. According to the first equation of (5.9), the magnetic field within the slab is

$$H(x) = Ae^{x/\lambda} + Be^{-x/\lambda} , \tag{5.11}$$

where A and B are constants. They are determined from the two requirements that at $x = d/2$ and at $x = -d/2$, the magnetic field equals the applied field, H_a. Hence,

$$H(x) = H_a \frac{\cosh(x/\lambda)}{\cosh(d/2\lambda)} . \tag{5.12}$$

The *average* value of the field within the slab is

$$\langle H \rangle = \frac{1}{d} \int_{-d/2}^{d/2} dx H_a \frac{\cosh(x/\lambda)}{\cosh(d/2\lambda)} = H_a \frac{2\lambda}{d} \tanh(d/2\lambda) . \tag{5.13}$$

This average value consists of the (external) applied field, H_a *plus* the magnetization induced in the slab, i.e.,

$$\langle H \rangle = H_a + 4\pi M . \tag{5.14}$$

When $d \gg \lambda$, $\langle H \rangle$ tends to zero, and therefore

$$M \to -\frac{H_a}{4\pi} , \quad d \gg \lambda . \tag{5.15}$$

The *susceptibility* is $-1/4\pi$, which means that a bulk superconductor is a perfect diamagnet. On the other hand, when $d \ll \lambda$, $\langle H \rangle$ tends to $H_a(1 - (d^2/12\lambda^2))$, and therefore

$$M \to -\Big(\frac{H_a}{4\pi}\Big)\Big(\frac{d^2}{12\lambda^2}\Big) . \tag{5.16}$$

We can estimate from this relation the critical magnetic field in the *special* case where the field is *parallel* to the slab. The critical field in this case is not the critical field of the material from which the slab is made, but it is the critical field of the material times the ratio λ/d,

$$H_{c\|} = \sqrt{12}\frac{\lambda}{d} H_c . \tag{5.17}$$

$* * *$ **exercise:** Explain intuitively the result (5.17) (by considering the current needed to screen the field). What will be the critical magnetic field *perpendicular* to the slab?

The Ginzburg-Landau (GL) theory. Ginzburg and Landau constructed a theory of superconductivity, by introducing a complex pseudo wave function ψ as an order parameter. The local density of the superconducting electrons is given by

$$n_s = |\psi(\mathbf{r})|^2 \ . \tag{5.18}$$

The order parameter obeys the Ginzburg-Landau equations,

$$\frac{1}{2m}\Big(-i\boldsymbol{\nabla} - \frac{e}{c}\mathbf{A}\Big)^2\psi + \beta|\psi|^2\psi = -\alpha\psi \ , \tag{5.19}$$

and the equation for the supercurrent density $\mathbf{J}_s$,

$$\mathbf{J}_s = -i\frac{e}{2m}\Big(\psi^*\boldsymbol{\nabla}\psi - \psi\boldsymbol{\nabla}\psi^*\Big) - \frac{e^2}{mc}|\psi|^2\mathbf{A} \ . \tag{5.20}$$

Here, α and β are (temperature-dependent) parameters, and $\mathbf{A}$ is the vector potential which represents the action of a constant magnetic field, $\mathbf{H} = \boldsymbol{\nabla} \times \mathbf{A}$. Note that if we write the complex order parameter in the form

$$\psi = |\psi|e^{i\phi} \ , \tag{5.21}$$

then the supercurrent is related to the*gradient* of the phase,

$$\mathbf{J}_s = \frac{e}{m}|\psi|^2\Big(\boldsymbol{\nabla}\phi - \frac{e}{c}\mathbf{A}\Big) \ . \tag{5.22}$$

The GL theory introduces a length,

$$\xi = \frac{1}{|2m\alpha|} \ , \tag{5.23}$$

which characterizes the distance over which $\psi(\mathbf{r})$ can vary. Near T_c ξ diverges as $(T_c - T)^{-1/2}$, since α vanishes as $(T - T_c)$. Thus, superconductivity is described by two lengths, the coherence length ξ and the penetration length λ. The ratio of these two lengths,

$$\kappa = \frac{\lambda}{\xi} \ , \tag{5.24}$$

is approximately temperature-independent. Type II superconductors are those for which $\kappa > 1/\sqrt{2}$. The GL equations can be derived from the free energy density, f, which takes the form

$$f = f_{n0} + \alpha|\psi|^2 + \frac{\beta}{2}|\psi|^4 + \frac{1}{2m}\Big|\Big(-i\boldsymbol{\nabla} - \frac{e}{c}\mathbf{A}\Big)\psi\Big|^2 + \frac{H^2}{8\pi} \ , \tag{5.25}$$

by minimizing the free energy with respect to the *complex* order parameter.

In a bulk superconductor, and in the absence of the magnetic field ($\mathbf{A} = 0$) we can take the order parameterψ to be real, since in this case all the coefficient of the differential equation are real. In a homogenous bulk superconductor we do not expect any spatial variation, and therefore Eq. (5.19) takes the form

$$\alpha\psi + \beta\psi^3 = 0 \ . \tag{5.26}$$

This has two solutions: either the bulk is simply not a superconductor, i.e., $\psi = 0$, or

$$\psi^2 \equiv \psi_\infty^2 = -\frac{\alpha}{\beta} \ . \tag{5.27}$$

This result leads to the identification of the temperature dependence of the coefficient α,

$$\alpha \propto -T_c + T \ . \tag{5.28}$$

Namely, for temperatures below the transition temperature T_c where α is negative, the superconducting order parameter is non zero. It vanishes (continuously) at $T = T_c$, and then system phase transforms into the normal state. Since this happens continuously (by construction) the GL equation describes properly the second order phase transition. Note that this argument ignores the temperature dependence of the other coefficient β. It assumes that β depends only weakly on the temperature.

Note that the superconducting free energy density, $\alpha|\psi|^2 + \frac{\beta}{2}|\psi|^4$ is zero when ψ is zero, and is $-|\alpha|^2/(2\beta$ when $|\psi|^2 = -\alpha/\beta$, namely, it is lower in the superconducting state.

Let us now assume that the superconductor occupies only half of the space, $x > 0$ (no fields are applied). Then the GL equation becomes

$$\alpha\psi + \beta\psi^3 - \frac{1}{2m}\frac{\partial^2\psi}{\partial x^2} = 0 \ . \tag{5.29}$$

Denoting

$$\psi = \psi_\infty f = (\sqrt{|\alpha|/\beta})f \ , \tag{5.30}$$

(ψ_∞ is the full value of the order parameter in the bulk superconductor), this equation becomes

$$\frac{1}{2m|\alpha|}\frac{d^2 f}{dx^2} + f - f^3 \equiv \xi^2\frac{d^2 f}{dx^2} + f - f^3 = 0 \ , \tag{5.31}$$

where we have used the definition (5.23) for the superconducting coherence length ξ. It is quite straightforward to solve Eq. (5.31). Denoting $f' \equiv df/dx$, and $f'' \equiv d^2f/dx^2$, we multiply Eq. (5.31) by f'. Then, $ff' = (1/2)df^2/dx$, $f^3f' = (1/4)df^4/dx$, and $f'f'' = (1/2)df'^2/dx$, and hence

$$\frac{d}{dx}\Big(\xi^2 f'^2 + f^2 - \frac{1}{2}f^4\Big) = 0 \ . \tag{5.32}$$

This implies that the combination of terms within the brackets do not depend on x, and consequently their value is the same as for $x \to \infty$, i.e., $f = 1$ [see Eq. (5.30)]. Namely,

$$\begin{aligned} &\xi^2 f'^2 + f^2 - \frac{1}{2}f^4 = \frac{1}{2} \Rrightarrow \frac{1}{2}(1 - f^2(x))^2 = \xi^2 f'^2(x) \\ &\Rrightarrow \frac{df}{1 - f^2} = \frac{dx}{\sqrt{2}\xi} \Rrightarrow f(x) = \tanh(\frac{x}{\sqrt{2}\xi}) \ . \end{aligned} \tag{5.33}$$

We see that near the boundary, the order parameter decays to zero over a scale length of order ξ.

$* * *$ **exercise:** Find and discuss the order parameter of a superconducting slab of width d, placed normal to the x axis (the slab is infinite along the y and the z directions). Discuss in particular the cases $d \ll \xi$ and $d \gg \xi$.

The critical current. There are certain cases in which one can assume that the absolute value of the order parameter, $|\psi|$, *does not* vary spatially, however, its phase ϕ, does [see Eq. (5.21)]. This occurs when the spatial change in $|\psi|$ has to occur over distances far smaller than ξ, and hence will cost too much kinetic energy. For example, if $|\psi|$ has to change over the width d of a thin film, its change will be of order $x/\xi \simeq d/\xi \ll 1$. In such cases the GL equations, [see Eqs. (5.22)) and (5.25)] take the form

$$\begin{aligned} \mathbf{J}_s &= \frac{e}{m}|\psi|^2(\boldsymbol{\nabla}\phi - \frac{e}{c}\mathbf{A}) \equiv e|\psi|^2\mathbf{v}_s \ , \\ f &= f_{n0} + \alpha|\psi|^2 + \frac{\beta}{2}|\psi|^4 + |\psi|^2\frac{mv_s^2}{2} + \frac{H^2}{8\pi} \ . \end{aligned} \tag{5.34}$$

In a very thin film or wire, of thickness $d \ll \lambda$, one may neglect the magnetic energy density $H^2/(8\pi)$ as compared to the kinetic energy (the latter is of order λ^2, and the former of order d^2). The super conducting free energy density is then

$$f_s = \alpha|\psi|^2 + \frac{\beta}{2}|\psi|^4 + |\psi|^2\frac{mv_s^2}{2} \ . \tag{5.35}$$

minimizing it with respect to $|\psi|^2$, we find

$$|\psi|^2 = -\frac{\alpha + mv_s^2/2}{\beta} = \frac{|\alpha|}{\beta}\Big(1 - \frac{mv_s^2}{2|\alpha|}\Big) = \psi_\infty^2\Big(1 - (\xi m v_s)^2\Big) \ . \tag{5.36}$$

The supercurrent is then

$$J_s = e\psi_\infty^2 v_s\Big(1 - (\xi m v_s)^2\Big) \ . \tag{5.37}$$

We see that the supercurrent vanishes when $v_s = 0$ and $v_s = 1/(m\xi)$. Its maximal value (found below) is the maximal supercurrent that the system can carry. The value of v_s at the maximum is obtained by minimizing J_s with respect to v_s, yielding $mv_s^2/2 = |\alpha|/3$. Inserting this value into Eq. (5.36), we find

$$|\psi|^2 = \frac{2}{3}\psi_\infty^2 \ , \tag{5.38}$$

namely, in the presence of current, the superconducting order parameter is reduced as compared to its value in the bulk (and in the absence of a current). The critical current, J_c, is given by inserting the results of the minimization into the first of Eqs. (5.34),

$$J_c = e\Psi_\infty^2 \frac{2}{3}\Big(\frac{2}{3}\frac{|\alpha|}{m}\Big)^{1/2} \ . \tag{5.39}$$

Note that the critical current vanishes at the transition temperature T_c, where $\alpha = 0$.

The upper critical field, H_{c2}. Let now consider how superconductivity is nucleated in the bulk, in the presence of a magnetic field $\mathbf{H}$ (along the z direction). We use the gauge

$$a_y = Hx \ , \tag{5.40}$$

and ignore in the GL differential equation the cubic term, since the system is only barely superconducting and the order parameter is therefore small. Then

$$\Big(-\nabla^2 + \frac{4\pi i}{\Phi_0}Hx\frac{\partial}{\partial y} + \Big(\frac{2\pi H}{\Phi_0}\Big)^2 x^2\Big)\psi = \frac{1}{\xi^2}\psi \ , \tag{5.41}$$

where the superconducting flux quantum is

$$\Phi_0 = \frac{\pi c}{e} \ . \tag{5.42}$$

By substituting

$$\psi(\mathbf{r}) = e^{ik_y y + ik_z z} f(x) \ , \tag{5.43}$$

we find

$$-\frac{\partial^2 f}{\partial x^2}+\Big(\frac{2\pi H}{\Phi_0}\Big)^2(x-x_0)^2 f=\Big(\frac{1}{\xi^2}-k_z^2\Big)f\ , \quad x_0=\frac{k_y\Phi_0}{2\pi H}\ . \tag{5.44}$$

The problem now is the same as the Schrödinger equation for an harmonic oscillator, with the eigenvalues

$$\epsilon_n=(n+\frac{1}{2})\omega_c=(n+\frac{1}{2})\Big(\frac{eH}{mc}\Big)\ . \tag{5.45}$$

In our case, however, the energy is given by the term on the right hand side of Eq. (5.44). The *maximal* magnetic field (corresponding to the lowest value of n, i.e., $n=0$) which the system can support is therefore

$$H=\frac{\Phi_0}{2\pi}\Big(\frac{1}{\xi^2}-k_z^2\Big)\ . \tag{5.46}$$

The truly maximal field is obtained for $k_z=0$ (i.e., no variation of the order parameter along the axis parallel to the field). Hence

$$H_{c2}=\frac{\Phi_0}{2\pi\xi^2}\ . \tag{5.47}$$

Note that this field vanishes at the superconducting transition temperature.

<u>The BCS Hamiltonian.</u> The 'reduced' Hamiltonian, which is assumed to include all interactions important for superconductivity, reads

$$\mathcal{H}=\sum_{\mathbf{k}\sigma}\zeta_k n_{\mathbf{k}\sigma}+\sum_{\mathbf{k}\mathbf{k}'}V_{\mathbf{k}\mathbf{k}'}c^\dagger_{\mathbf{k}\uparrow}c^\dagger_{-\mathbf{k}\downarrow}c_{-\mathbf{k}'\downarrow}c_{\mathbf{k}'\uparrow}\ . \tag{5.48}$$

Here, $c^\dagger_{\mathbf{k}\sigma}$ is the operator that creates an electron at state of wave vector $\mathbf{k}$ and spin σ, $c_{\mathbf{k}\sigma}$ is the operator that destroys such a state (these operators obey the anti commutation fermionic relations), $n_{\mathbf{k}\sigma}$ is the number operator of electrons in the state $\mathbf{k}$ with spin σ

$$n_{\mathbf{k}\sigma}=c^\dagger_{\mathbf{k}\sigma}c_{\mathbf{k}\sigma}\ , \tag{5.49}$$

and $V_{\mathbf{k}\mathbf{k}'}$ is the pairing interaction: it destroys a pair of electrons of opposite spins and momenta, and creates another pair of opposite spins and momenta. Finally, ζ_k is the single electron energy, measured from the Fermi energy.

The BCS Hamiltonian may be solved using the mean-field approximation, which we have already encountered in the discussion of the Heisenberg exchange interaction. We replace

$$c^\dagger_{\mathbf{k}\uparrow}c^\dagger_{-\mathbf{k}\downarrow}c_{-\mathbf{k}'\downarrow}c_{\mathbf{k}'\uparrow}\Longrightarrow\langle c_{-\mathbf{k}'\downarrow}c_{\mathbf{k}'\uparrow}\rangle c^\dagger_{\mathbf{k}\uparrow}c^\dagger_{-\mathbf{k}\downarrow}+\langle c^\dagger_{\mathbf{k}\uparrow}c^\dagger_{-\mathbf{k}\downarrow}\rangle c_{-\mathbf{k}'\downarrow}c_{\mathbf{k}'\uparrow}-\langle c_{-\mathbf{k}'\downarrow}c_{\mathbf{k}'\uparrow}\rangle\langle c^\dagger_{\mathbf{k}\uparrow}c^\dagger_{-\mathbf{k}\downarrow}\rangle\ , \tag{5.50}$$

and denote

$$b_{\mathbf{k}} = \langle c_{-\mathbf{k}\downarrow} c_{\mathbf{k}\uparrow} \rangle \ , \quad b^*_{\mathbf{k}} = \langle c^\dagger_{\mathbf{k}\uparrow} c^\dagger_{-\mathbf{k}\downarrow} \rangle \ . \tag{5.51}$$

The model Hamiltonian, $\mathcal{H}_M$, that results reads

$$\mathcal{H}_M = \sum_{\mathbf{k}\sigma} \zeta_k n_{\mathbf{k}\sigma} - \sum_{\mathbf{k}} (\Delta_{\mathbf{k}} c^\dagger_{\mathbf{k}\uparrow} c^\dagger_{-\mathbf{k}\downarrow} + \Delta^*_{\mathbf{k}} c_{-\mathbf{k}\downarrow} c_{\mathbf{k}\uparrow} - \Delta_{\mathbf{k}} b^*_{\mathbf{k}}) \ , \tag{5.52}$$

where we have defined

$$\Delta_{\mathbf{k}} = -\sum_{\mathbf{k}'} V_{\mathbf{k}\mathbf{k}'} b_{\mathbf{k}'} \equiv \sum_{\mathbf{k}'} V_{\mathbf{k}\mathbf{k}'} \langle c_{-\mathbf{k}\downarrow} c_{\mathbf{k}'\uparrow} \rangle \ . \tag{5.53}$$

This model Hamiltonian is diagonalized by the Bogoliubov transformation:

$$c_{\mathbf{k}\uparrow} = u^*_{\mathbf{k}} \gamma_{\mathbf{k}0} + v_{\mathbf{k}} \gamma^\dagger_{\mathbf{k}1} \ , \quad c^\dagger_{-\mathbf{k}\downarrow} = -v^*_{\mathbf{k}} \gamma_{\mathbf{k}0} + u_{\mathbf{k}} \gamma^\dagger_{\mathbf{k}1} \ . \tag{5.54}$$

$* * *$ **exercise:** Check that in order for the new operators $\gamma_{\mathbf{k}0}$ and $\gamma_{\mathbf{k}1}$ to be fermions (that is, to obey the anti commutation relations), it suffices that $|u_{\mathbf{k}}|^2 + |v_{\mathbf{k}}|^2 = 1$.

$* * *$ **exercise:** Show that

$$\gamma_{\mathbf{k}0} = u_{\mathbf{k}} c_{\mathbf{k}\uparrow} - v_{\mathbf{k}} c^\dagger_{-\mathbf{k}\downarrow} \quad \gamma^\dagger_{\mathbf{k}1} = u^*_{\mathbf{k}} c^\dagger_{-\mathbf{k}\downarrow} + v^*_{\mathbf{k}} c_{\mathbf{k}\uparrow} \ . \tag{5.55}$$

Explain the meaning of these operators.

The next step is to insert Eqs. (5.54) into the model Hamiltonian (5.52). This procedure gives

$$\begin{aligned} \mathcal{H}_M = & \sum_{\mathbf{k}} \zeta_k \Big[(|u_{\mathbf{k}}|^2 - |v_{\mathbf{k}}|^2)(\gamma^\dagger_{\mathbf{k}0}\gamma_{\mathbf{k}0} + \gamma^\dagger_{\mathbf{k}1}\gamma_{\mathbf{k}1}) + 2|v_{\mathbf{k}}|^2 + 2u^*_{\mathbf{k}} v^*_{\mathbf{k}} \gamma_{\mathbf{k}1}\gamma_{\mathbf{k}0} + 2u_{\mathbf{k}} v_{\mathbf{k}} \gamma^\dagger_{\mathbf{k}0}\gamma^\dagger_{\mathbf{k}1} \Big] \\ & + \sum_{\mathbf{k}} \Big[(\Delta_{\mathbf{k}} u_{\mathbf{k}} v^*_{\mathbf{k}} + \Delta^*_{\mathbf{k}} u^*_{\mathbf{k}} v_{\mathbf{k}})(\gamma^\dagger_{\mathbf{k}0}\gamma_{\mathbf{k}0} + \gamma^\dagger_{\mathbf{k}1}\gamma_{\mathbf{k}1} - 1) + (\Delta_{\mathbf{k}} v^{*\,2}_{\mathbf{k}} - \Delta^*_{\mathbf{k}} u^{*\,2}_{\mathbf{k}})\gamma_{\mathbf{k}1}\gamma_{\mathbf{k}0} \\ & \quad + (\Delta^*_{\mathbf{k}} v^2_{\mathbf{k}} - \Delta_{\mathbf{k}} u^2_{\mathbf{k}})\gamma^\dagger_{\mathbf{k}0}\gamma^\dagger_{\mathbf{k}1} + \Delta_{\mathbf{k}} b^*_{\mathbf{k}} \Big] \ . \end{aligned} \tag{5.56}$$

Now we see that when the condition

$$2\zeta_k u_{\mathbf{k}} v_{\mathbf{k}} + \Delta^*_{\mathbf{k}} v^2_{\mathbf{k}} - \Delta_{\mathbf{k}} u^2_{\mathbf{k}} = 0 \tag{5.57}$$

is satisfied, the Hamiltonian becomes diagonal. Moreover, upon multiplying this condition by $\Delta^*_{\mathbf{k}}/u^2_{\mathbf{k}}$ we obtain

$$\frac{\Delta^*_{\mathbf{k}} v_{\mathbf{k}}}{u_{\mathbf{k}}} = (\zeta^2_k + |\Delta_{\mathbf{k}}|^2)^{1/2} - \zeta_k \equiv E_{\mathbf{k}} - \zeta_k = \text{real} \ . \tag{5.58}$$

This means that

$$\left|\frac{v_{\mathbf{k}}}{u_{\mathbf{k}}}\right| = \frac{E_{\mathbf{k}} - \zeta_k}{|\Delta_{\mathbf{k}}|} \implies |v_{\mathbf{k}}|^2 = 1 - |u_{\mathbf{k}}|^2 = \frac{1}{2}\Big(1 - \frac{\zeta_k}{E_{\mathbf{k}}}\Big) \ . \tag{5.59}$$

Inserting all these results into the model Hamiltonian (5.56), we finally obtain

$$\mathcal{H}_M = \sum_{\mathbf{k}} (\zeta_k - E_{\mathbf{k}} + \Delta_{\mathbf{k}} b^*_{\mathbf{k}}) + \sum_{\mathbf{k}} E_{\mathbf{k}} (\gamma^\dagger_{\mathbf{k}0}\gamma_{\mathbf{k}0} + \gamma^\dagger_{\mathbf{k}1}\gamma_{\mathbf{k}1}) \ . \tag{5.60}$$

The first term here is a constant, and the second is just the Hamiltonian of free fermions, with excitation energy $E_{\mathbf{k}}$. We see from Eq. (5.58) that the excitation spectrum has a gap, of magnitude $|\Delta_{\mathbf{k}}|$. Another important point is that $\Delta_{\mathbf{k}}$ itself, as given in Eq. (5.53), becomes

$$\Delta_{\mathbf{k}} = -\sum_{\mathbf{k}'} V_{\mathbf{k}\mathbf{k}'} \langle c_{-\mathbf{k}'\downarrow} c_{\mathbf{k}\uparrow} \rangle = -\sum_{\mathbf{k}'} V_{\mathbf{k}\mathbf{k}'} u^*_{\mathbf{k}'} v_{\mathbf{k}'} \langle 1 - \gamma^*_{\mathbf{k}'0}\gamma_{\mathbf{k}'0} - \gamma^\dagger_{\mathbf{k}'1}\gamma_{\mathbf{k}'1} \rangle \ , \tag{5.61}$$

where, using Eq. (5.58)

$$u^*_{\mathbf{k}} v_{\mathbf{k}} = \frac{\Delta_{\mathbf{k}}}{2E_{\mathbf{k}}} \ . \tag{5.62}$$

Since the model Hamiltonian is diagonal, we know that

$$\langle \gamma^\dagger_{\mathbf{k}0}\gamma_{\mathbf{k}0} \rangle = \langle \gamma^\dagger_{\mathbf{k}1}\gamma_{\mathbf{k}1} \rangle = \frac{1}{e^{\beta E_{\mathbf{k}}} + 1} \equiv f(E_{\mathbf{k}}) \ . \tag{5.63}$$

Using this in Eqs. (5.61) and (5.62) yields the *gap equation*,

$$\Delta_{\mathbf{k}} = -\sum_{\mathbf{k}'} V_{\mathbf{k}\mathbf{k}'} \frac{\Delta_{\mathbf{k}'}}{2E_{\mathbf{k}'}} \tanh\frac{\beta E_{\mathbf{k}'}}{2} \ . \tag{5.64}$$

In particular, with the simplifying BCS assumption, $V_{\mathbf{k}\mathbf{k}'} = -V$, we find

$$\frac{1}{V} = \frac{1}{2}\sum_{\mathbf{k}} \frac{\tanh(\beta E_k/2)}{E_k} \ . \tag{5.65}$$

One should note that the BCS assumption about the pairing potential V cannot hold over the entire Brillouin zone; it is expected to be valid in a narrow range of energies about the Fermi energy. That narrow energy range is limited by a certain energy, ω_c. When the pairing potential is due to the electron-phonon interaction, then ω_c is a typical phonon energy. Taking all this into account, we may convert Eq. (5.65) into the famous BCS form,

$$\frac{1}{N(0)V} = \int_0^{\omega_c} d\zeta \frac{\tanh\frac{1}{2}(\beta\sqrt{\zeta^2 + \Delta^2})}{\sqrt{\zeta^2 + \Delta^2}} \ , \tag{5.66}$$

where $N(0)$ is the density of states at the Fermi level. This equation determines the temperature dependence of the gap Δ, and in particular, it yields the transition temperature into the superconducting state. At the transition temperature T_c, Δ vanishes, and therefore we have

$$\frac{1}{N(0)V} = \int_0^{\omega_c} d\zeta \frac{\tanh\frac{1}{2}(\beta_c\zeta)}{\zeta} \quad \Longrightarrow k_B T_c = 1.13\omega_c e^{-1/N(0)V} \ . \tag{5.67}$$

Bibliography

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