Statistical Mechanics - Class Exercise 3

April 26, 2022

The basic integrals for ideal gas

$$N = \sum_r f(\epsilon_r - \mu) = \int_0^\infty g(\epsilon) f(\epsilon - \mu) \, d\epsilon$$

$$E = \sum_r \epsilon_r f(\epsilon_r - \mu) = \int_0^\infty g(\epsilon) \epsilon f(\epsilon - \mu) \, d\epsilon$$

$$\boldsymbol{Z} = \prod_r \mathcal{Z}^{(r)}, \quad \mathcal{Z}^{(r)} = \sum_n e^{-\beta(\epsilon_r - \mu)n_r} = \left(1 \pm e^{-\beta(\epsilon_r - \mu)}\right)^{\pm 1}$$

$$\ln(\boldsymbol{Z}) = \pm \sum_r \ln\left(1 \pm e^{-\beta(\epsilon_r - \mu)}\right) = \{\text{Integration by parts}\} = \beta \int_0^\infty \mathcal{N}(\epsilon) f(\epsilon - \mu) \, d\epsilon$$

$$P = \frac{1}{\beta} \frac{\partial \ln(\boldsymbol{Z})}{\partial V} = \frac{1}{\beta} \frac{\ln(\boldsymbol{Z})}{V} = \frac{1}{V} \int_0^\infty \mathcal{N}(\epsilon) f(\epsilon - \mu) \, d\epsilon$$

Exercise 3336 - Condensation for general dispersion

An ideal Bose gas consists of particles that have the dispersion relation $\epsilon = c|p|^s$ with $s > 0$. The gas is contained in a box that has volume V in d dimensions. The gas is maintained in a uniform temperature T.

1. Calculate the single particle density of states.
2. Find a condition involving s and d for the existence of Bose-Einstein condensation. In particular relate to relativistic ($s = 1$) and nonrelativistic ($s = 2$) particles in two dimensions.
3. Find the dependence of the number of particles N on the chemical potential μ.
4. Find the dependence of the total energy E on the chemical potential, and show how the pressure P is obtained from this result.
5. Find an expression for the heat capacity C_v. Show how this result can be expressed using N in the limit of infinite temperature.
6. Repeat item 1 for relativistic gas whose particles have finite mass such that their dispersion relation is $\epsilon = \sqrt{m^2c^4 + c^2p^2}$.
7. Consider a relativistic gas in $2D$. Find expressions for N and E and P. Should one expect Bose-Einstein condensation?

Answer

1. We have the dispersion relation $\epsilon = c|p|^s$

$$\rightarrow |p| = \left(\frac{\epsilon}{c}\right)^{\frac{1}{s}}$$

$$\mathcal{N}(\epsilon) = \int_{H<\epsilon} \frac{d^dxd^dp}{(2\pi)^d} = \frac{V}{(2\pi)^d}\int\cdots\int_{\sqrt{p_1^2+\cdots+p_d^2}\leq\left(\frac{\epsilon}{c}\right)^{\frac{1}{s}}} d^dp = \frac{V}{(2\pi)^d}\Omega_d\int_0^{\left(\frac{\epsilon}{c}\right)^{\frac{1}{s}}} p^{d-1}dp = \frac{V}{(2\pi)^d}\frac{\Omega_d}{d}\left(\frac{\epsilon}{c}\right)^{\frac{d}{s}}$$

$$\mathcal{N}(\epsilon) = \frac{1}{d}\frac{V}{(2\pi)^d}\frac{\Omega_d}{c^{\frac{d}{s}}}\epsilon^{\frac{d}{s}}$$

$$g(\epsilon) = \frac{\partial\mathcal{N}(\epsilon)}{\partial\epsilon} = \frac{1}{s}\frac{V}{(2\pi)^d}\frac{\Omega_d}{c^{\frac{d}{s}}}\epsilon^{\frac{d}{s}-1}$$

$$\rightarrow \mathcal{N}(\epsilon) = \frac{s}{d}g(\epsilon)\epsilon$$

2. We define $\alpha = \frac{d}{s}, c = \frac{1}{(2\pi)^d}\frac{\Omega_d}{sc^{\frac{d}{s}}}$
In general

$$N = \int_0^\infty g(\epsilon) f(\epsilon-\mu) d\epsilon = cV\int_0^\infty \frac{\epsilon^{\alpha-1}}{e^{\beta(\epsilon-\mu)}-1}d\epsilon$$

For Bose-Einstein condensation we need that the integral will converge, and this happen for $\alpha > 1 \rightarrow d > s$.
For nonrelativistic ($s = 2$) particles in 2D $d = s$, so the system will not exhibit BEC.
For relativistic ($s = 1$) particles in 2D $d > s$, so the system can exhibit BEC.
3. Define $z = e^{\beta\mu}$

$$N = cV\int_0^\infty \frac{\epsilon^{\alpha-1}}{e^{\beta(\epsilon-\mu)}-1}d\epsilon = \left\{\begin{array}{c} x = \beta\epsilon \\ Tdx = d\epsilon \end{array}\right\} = cVT^\alpha\int_0^\infty \frac{x^{\alpha-1}}{\frac{1}{z}e^x-1}dx = cVT^\alpha F_\alpha(\beta\mu)$$

Where

$$F_\alpha(\beta\mu) = \Gamma(\alpha) Li_\alpha(z), \quad Li_\alpha(z) = \sum_{\ell=1}^\infty \frac{z^\ell}{\ell^\alpha}$$

4. To calculate the total energy E:

$$E = \int_0^\infty g(\epsilon)\epsilon f(\epsilon-\mu) d\epsilon = cV\int_0^\infty \frac{\epsilon^\alpha}{e^{\beta(\epsilon-\mu)}-1}d\epsilon = cVT^{\alpha+1}F_{\alpha+1}(\beta\mu)$$

In the other side

$$\ln(Z) = \beta\int_0^\infty \mathcal{N}(\epsilon) f(\epsilon-\mu) d\epsilon = \frac{\beta}{\alpha}\int_0^\infty g(\epsilon)\epsilon f(\epsilon-\mu) d\epsilon = \frac{\beta}{\alpha}cVT^{\alpha+1}F_{\alpha+1}(\beta\mu) = \frac{\beta}{\alpha}E$$

We know that

$$P = \frac{1}{\beta}\frac{\partial\ln(Z)}{\partial V} = \frac{1}{\beta}\frac{\ln(Z)}{V} = \frac{1}{\alpha}\frac{E}{V}$$

5. The heat capacity C_v.

$$C_v = \left.\frac{\partial E}{\partial T}\right|_{V,N}$$

In the limit of infinite temperature can use Boltzmann approximation; in this limit $f(\epsilon-\mu)=e^{-\beta(\epsilon-\mu)}$

$$N=cV\int_0^\infty \epsilon^{\alpha-1}e^{-\beta(\epsilon-\mu)}d\epsilon=cVe^{\beta\mu}T^\alpha\int_0^\infty x^{\alpha-1}e^{-x}dx=cVe^{\beta\mu}T^\alpha\Gamma(\alpha)$$

$$E=cV\int_0^\infty \epsilon^{\alpha}e^{-\beta(\epsilon-\mu)}d\epsilon=cVe^{\beta\mu}T^{\alpha+1}\Gamma(\alpha+1)=cVe^{\beta\mu}T^{\alpha+1}\alpha\Gamma(\alpha)=\alpha NT$$

$$\to C_v=\left.\frac{\partial E}{\partial T}\right|_{V,N}=\alpha N$$

6. For relativistic gas with dispersion relation is $\epsilon=\sqrt{m^2c^4+c^2p^2}$.

$$\to |p|=\sqrt{\left(\frac{\epsilon}{c}\right)^2-m^2c^2}$$

$$\mathcal{N}(\epsilon)=\int_{H<\epsilon}\frac{d^dxd^dp}{(2\pi)^d}=\frac{V}{(2\pi)^d}\int\cdots\int_{\sqrt{p_1^2+\cdots+p_d^2}\leq\sqrt{\left(\frac{\epsilon}{c}\right)^2-m^2c^2}}d^dp=\frac{V}{(2\pi)^d}\Omega_d\int_0^{\sqrt{\left(\frac{\epsilon}{c}\right)^2-m^2c^2}}p^{d-1}dp$$
$$=\frac{V}{(2\pi)^d}\frac{\Omega_d}{d}\left(\left(\frac{\epsilon}{c}\right)^2-m^2c^2\right)^{\frac{d}{2}}$$

$$g(\epsilon)=\frac{\partial\mathcal{N}(\epsilon)}{\partial\epsilon}=\frac{V}{(2\pi)^d}\Omega_d\left(\left(\frac{\epsilon}{c}\right)^2-m^2c^2\right)^{\frac{d}{2}-1}\frac{\epsilon}{c^2}$$

7. For relativistic gas in $2D$ the momentum $|p|\sim\epsilon\to\alpha=\frac{d}{s}=2$

$$\mathcal{N}(\epsilon)=\frac{1}{2}\frac{V}{2\pi c^2}\left(\epsilon^2-m^2c^4\right)$$

$$g(\epsilon)=\frac{\partial\mathcal{N}(\epsilon)}{\partial\epsilon}=\frac{V}{2\pi c^2}\epsilon$$

$$N=\frac{V}{2\pi c^2}\int_{mc^2}^\infty\frac{\epsilon}{e^{\beta(\epsilon-\mu)}-1}d\epsilon=\left\{\epsilon'=\epsilon-mc^2\right\}=\frac{V}{2\pi c^2}\int_0^\infty\frac{\epsilon'+mc^2}{e^{\beta(\epsilon'+mc^2-\mu)}-1}d\epsilon'=$$
$$=\frac{V}{2\pi c^2}T^2F_2\left(\frac{\mu-mc^2}{T}\right)+\frac{Vmc^2}{2\pi c^2}TF_1\left(\frac{\mu-mc^2}{T}\right)$$
$$N=\frac{V}{2\pi c^2}\left[T^2F_2\left(\frac{\mu-mc^2}{T}\right)+mc^2TF_1\left(\frac{\mu-mc^2}{T}\right)\right]$$

In the same way

$$E=\frac{V}{2\pi c^2}\int_{mc^2}^\infty\frac{\epsilon^2}{e^{\beta(\epsilon-\mu)}-1}d\epsilon=\frac{V}{2\pi c^2}\int_0^\infty\frac{\epsilon'^2+2\epsilon'mc^2+m^2c^4}{e^{\beta(\epsilon'+mc^2-\mu)}-1}d\epsilon'$$
$$E=\frac{V}{2\pi c^2}\left[T^3F_3\left(\frac{\mu-mc^2}{T}\right)+2mc^2T^2F_2\left(\frac{\mu-mc^2}{T}\right)+m^2c^4TF_1\left(\frac{\mu-mc^2}{T}\right)\right]$$

For P we need to calculate $\ln(Z)$

$$\ln(Z) = \beta\int_0^\infty \mathcal{N}(\epsilon) f(\epsilon-\mu)\, d\epsilon = \beta\frac{V}{4\pi c^2}\int_{mc^2}^\infty \frac{\left(\epsilon^2 - m^2c^4\right)}{e^{\beta(\epsilon-\mu)}-1} d\epsilon$$

$$= \beta\frac{V}{4\pi c^2}\int_0^\infty \frac{\epsilon'^2 + 2\epsilon' mc^2}{e^{\beta(\epsilon'+mc^2-\mu)}-1} d\epsilon' = \beta\frac{V}{4\pi c^2}\left[T^3F_3\left(\frac{\mu-mc^2}{T}\right) + 2mc^2T^2F_2\left(\frac{\mu-mc^2}{T}\right)\right]$$

$$P = \frac{1}{\beta}\frac{\ln(Z)}{V} = \frac{1}{4\pi c^2}\left[T^3F_3\left(\frac{\mu-mc^2}{T}\right) + 2mc^2T^2F_2\left(\frac{\mu-mc^2}{T}\right)\right]$$

for get BEC we need that $N = \int_0^\infty g(\epsilon) f(\epsilon-\mu)\, d\epsilon$ will be finite for $\mu \to 0$

$$N = \frac{V}{2\pi c^2}\int_0^\infty \frac{\epsilon' + mc^2}{e^{\beta(\epsilon'+mc^2-\mu)}-1} d\epsilon' = \frac{V}{2\pi c^2}\int_0^\infty \frac{\epsilon' + mc^2}{e^{\beta(\epsilon'-\mu')}-1} d\epsilon'$$

For $\mu \to mc^2 \Leftrightarrow \mu' \to 0$ we get that $\int_0^\infty \frac{mc^2}{e^{\beta(\epsilon'-\mu')}-1} d\epsilon'$ does not converge and so we can't expect to get BEC.
In another way we can see that for $p \ll 1$

$$\epsilon = \sqrt{m^2c^4 + c^2p^2} \approx mc^2 + \frac{p^2}{2m} \to \alpha = 1$$

Exercise 3021 - Spin 1 bosons in 3D box with Zeeman interaction

N Bosons that have mass m and spin 1 are placed in a box that has volume V. A magnetic field B is applied, such that the interaction is $-\gamma B S_z$, where $S_z = 1, 0, -1$, and γ is the gyromagnetic ratio. In items (3-6) assume the Boltzmann approximation for the occupation of the $S_z \neq 1$ states.

1. Find an equation for the condensation temperature T_c.
2. Find the condensation temperature $T_c(B)$ for $B = 0$ and for $B \to \infty$.
3. Find the critical B for condensation if T is set in the range of temperatures that has been defined in item (2).
4. Describe how $T_c(B)$ depends of B in a qualitatively manner. Find approximate expressions for moderate and large fields.
5. Find the condensate fraction as a function of T and B.
6. Find the heat capacity of the gas assuming large but finite field.

Answer

1. The Hamiltonian for one particle:

$$H = \frac{p^2}{2m} - \gamma B S_z$$

For $\alpha = \frac{d}{s} = \frac{3}{2}, c = \frac{1}{(2\pi)^d}\frac{\Omega_d}{sc^{\frac{d}{s}}} = \frac{(2m)^{\frac{3}{2}}}{(2\pi)^2}$ For spinless particles we get

$$N = cVT^{\frac{3}{2}}F_{\frac{3}{2}}(\beta\mu) = V\frac{2^{\frac{3}{2}}\left(\frac{mT}{2\pi}\right)^{\frac{3}{2}}}{(2\pi)^{\frac{1}{2}}}\Gamma\left(\frac{3}{2}\right) Li_{\frac{3}{2}}\left(e^{\beta\mu}\right) = \frac{V}{\lambda_T^3} Li_{\frac{3}{2}}\left(e^{\beta\mu}\right)$$

we can treat the particles of the system like three different spinless gasses with different S_z.

$$N=\sum f\left(\epsilon-\mu\right)=\sum f\left(\frac{p^2}{2m}-\gamma B-\mu\right)+\sum f\left(\frac{p^2}{2m}-\mu\right)+\sum f\left(\frac{p^2}{2m}+\gamma B-\mu\right)$$

We can define $\mu'=\mu+\gamma BS_z$ and get

$$\frac{N}{V}=\frac{1}{\lambda_T^3}\left(\mathrm{Li}_{3/2}\left(e^{\beta(\mu+\gamma B)}\right)+\mathrm{Li}_{3/2}\left(e^{\beta\mu}\right)+\mathrm{Li}_{3/2}\left(e^{\beta(\mu-\gamma B)}\right)\right)$$

The condition to condensation is that μ goes to the lowest energy, here this mean $\mu\to-\gamma B$ $(S_z=1)$

$$\frac{N}{V}=n=n_0+\frac{1}{\lambda_T^3}\left(\mathrm{Li}_{3/2}\left(1\right)+\mathrm{Li}_{3/2}\left(e^{-\beta\gamma B}\right)+\mathrm{Li}_{3/2}\left(e^{-2\beta\gamma B}\right)\right)$$

For $T=T_c$

$$n\approx\frac{1}{\lambda_T^3}\left(\zeta\left(\frac{3}{2}\right)+e^{-\beta\gamma B}+e^{-2\beta\gamma B}\right),\ \ \zeta\left(\frac{3}{2}\right)\approx 2.612$$

2. For $B=0$\frac{3}{2}\left(\frac{m}{2\pi}\right)^{\frac{3}{2}}

$$n=\frac{1}{\lambda_T^3}\left(\mathrm{Li}_{3/2}\left(1\right)+\mathrm{Li}_{3/2}\left(1\right)+\mathrm{Li}_{3/2}\left(1\right)\right)=\frac{3}{\lambda_T^3}\zeta\left(\frac{3}{2}\right)=3\left(\frac{m}{2\pi}\right)^{\frac{3}{2}}\zeta\left(\frac{3}{2}\right)T_c^{\frac{3}{2}}$$

$$\to T_c\left(B=0\right)=\frac{2\pi}{m}\left(\frac{n}{3\cdot 2.612}\right)^{\frac{2}{3}}$$

For $B\to\infty$ the occupation states are just $S_z=1$, we get $\mathrm{Li}_{3/2}\left(e^{-\beta\gamma B}\right)\to 0$

$$n=\left(\frac{m}{2\pi}\right)^{\frac{3}{2}}\zeta\left(\frac{3}{2}\right)T_c^{\frac{3}{2}}$$

$$\to T_c\left(B=\infty\right)=\frac{2\pi}{m}\left(\frac{n}{2.612}\right)^{\frac{2}{3}}$$

$$\frac{T_c\left(B=\infty\right)}{T_c\left(B=0\right)}=3^{\frac{2}{3}}\approx 2$$

3. Now we assume $B\neq 0,\ \gamma B\gg T$, so for $S_z\neq 1$ we can use the Boltzmann approximation.

$$n\approx\frac{1}{\lambda_T^3}\left(\mathrm{Li}_{3/2}\left(1\right)+e^{-\beta\gamma B}+e^{-2\beta\gamma B}\right)\approx\frac{1}{\lambda_T^3}\left(\zeta\left(\frac{3}{2}\right)+e^{-\beta\gamma B}\right)$$

$$\to B_c=-\frac{T}{\gamma}\ln\left(\lambda_T^3 n-\zeta\left(\frac{3}{2}\right)\right)$$

4. As B is increased, T_c rises until B_c is reached. At $B=B_c, T=T_c$ and the condensation occurs

$$n=\left(\frac{m}{2\pi}\right)^{\frac{3}{2}}T_c^{\frac{3}{2}}\left(\zeta\left(\frac{3}{2}\right)+e^{-\beta\gamma B}\right)$$

5. We get

$$n_0 = n - \frac{1}{\lambda_T^3}\left(\zeta\left(\frac{3}{2}\right) + e^{-\beta\gamma B}\right)$$

So

$$\frac{n_0}{n} = 1 - \frac{\left(\zeta\left(\frac{3}{2}\right) + e^{-\beta\gamma B}\right)}{n\lambda_T^3} = 1 - \frac{n\left(T,B\right)}{n}$$

6. For large but finite field $\gamma B \gg T$

$$\frac{E}{V} = \frac{3}{2}\frac{T}{\lambda_T^3}\left(\mathrm{Li}_{5/2}\left(1\right) + e^{-\beta\gamma B} + e^{-2\beta\gamma B}\right) \approx \frac{3}{2}\left(\frac{m}{2\pi}\right)^{\frac{3}{2}} T^{\frac{5}{2}}\left(\zeta\left(\frac{5}{2}\right) + e^{-\frac{\gamma B}{T}}\right)$$

$$\frac{C_V}{V} = \frac{\partial}{\partial T}\left(\frac{E}{V}\right) = \frac{3}{2}\frac{1}{\lambda_T^3}\left[\frac{5}{2}\zeta\left(\frac{5}{2}\right) + \left(\frac{5}{2} + \beta\gamma B\right)e^{-\beta\gamma B}\right] \approx \frac{3}{2}\frac{1}{\lambda_T^3}\left[\frac{5}{2}\zeta\left(\frac{5}{2}\right) + \beta\gamma B e^{-\beta\gamma B}\right]$$

Exercise 3745 - Fermions in a uniform gravitational field

Consider fermions of mass M and spin 1/2 in a gravitational field with constant acceleration g and at uniform temperature T. The density of the Fermions at zero height is $n(0) = n_0$. In item (3) assume that at zero height the fermions form a degenerate gas with Fermi energy ϵ_F^0 that is much larger compared with T.

1. Assume that the fermions behave as classical particles and find their density $n(h)$ as function of the height.
2. Assume $T = 0$. Find the local Fermi momentum $p_F(h)$ and the density $n(h)$ as function of the height.
3. Assume low temperatures. Estimate the height h_c such that for $h \gg h_c$ the fermions are non-degenerate.
4. In the latter case find $n(h)$ for $h \gg h_c$, given as before n_0 at zero height.

Answer

1. We look on a layer of gas in the high $z \in [h, h + \delta h]$ in a virtual box with size $L \times L \times \delta h$.
The partition function for one classical particles

$$\mathcal{Z}_1\left(\beta,h\right) = \underbrace{2}_{spin}\int_0^L\int_0^L\int_h^{h+\delta h} e^{-\beta\left(\frac{p^2}{2m}+mgz\right)}\frac{dxdydzd^3p}{\left(2\pi\right)^3} = \frac{2L^2}{\beta mg\lambda_T^3}\left(e^{-\beta mgh} - e^{-\beta mg(h+\delta h)}\right)$$

$$= \frac{2L^2}{\beta mg\lambda_T^3}e^{-\beta mgh}\left(1 - e^{-\beta mg\delta h}\right) \approx \frac{2L^2\delta h}{\lambda_T^3}e^{-\beta mgh}$$

The partition function with $L^2\delta h = V$:

$$\mathcal{Z}_N\left(\beta,h\right) = \frac{1}{N\left(h\right)!}\left(\frac{2V}{\lambda_T^3}e^{-\beta mgh}\right)^{N(h)}$$

The Free energy:

$$F\left(\beta,h\right) = -T\ln\mathcal{Z}_N = T\ln\left(N\left(h\right)!\right) - TN\left(h\right)\ln\left(\frac{2V}{\lambda_T^3}\right) + N\left(h\right)mgh$$

with Stirling's approximation

$$F(\beta,h)\approx -TN(h)-TN(h)\ln\left(\frac{2V}{N(h)\lambda_T^3}\right)+N(h)\,mgh$$

The chemical potential:

$$\mu(\beta,h)=\frac{\partial F(\beta,h)}{\partial N(h)}=-T\ln\left(\frac{2V}{N(h)\lambda_T^3}\right)+mgh=-T\ln\left(\frac{2}{n(h)\lambda_T^3}\right)+mgh$$

at zero height

$$\mu(0)=-T\ln\left(\frac{2}{n(0)\lambda_T^3}\right)$$

From chemical equilibrium $\mu(h)=\mu(0)$

$$-T\ln\left(\frac{2}{n(h)\lambda_T^3}\right)+mgh=-T\ln\left(\frac{2}{n_0\lambda_T^3}\right)$$

$$\beta mgh=\ln\left(\frac{n_0}{n(h)}\right)$$

$$n(h)=n_0e^{-\beta mgh}$$

2. At $T=0$ we have a degenerate gas when all the energy state up to ϵ_F are occupied. The number of states:

$$N(\beta,h,\mu)=\int_0^\infty g(\epsilon_p)\,f(\epsilon_p+mgh-\mu)\,d\epsilon_p$$

where for fermions of mass M and spin 1/2

$$g(\epsilon)=2\frac{V(2m)^{\frac{3}{2}}}{(2\pi)^2}\epsilon^{\frac{1}{2}}$$

In $T=0$ the occupation function is a step function with $\mu'(h)=\epsilon_F-mgh$, so we get

$$n(h)=\frac{(2m)^{\frac{3}{2}}}{3\pi^2}(\epsilon_F-mgh)^{\frac{3}{2}}$$

At zero hight

$$n(0)=n_0=\frac{(2m)^{\frac{3}{2}}}{3\pi^2}(\epsilon_F)^{\frac{3}{2}}$$

$$\epsilon_F=\frac{\left(3\pi^2n_0\right)^{\frac{2}{3}}}{2m}$$

$$\rightarrow n(h)=\frac{(2m)^{\frac{3}{2}}}{3\pi^2}\left(\frac{\left(3\pi^2n_0\right)^{\frac{2}{3}}}{2m}-mgh\right)^{\frac{3}{2}}$$

For the Fermi momentum

$$\epsilon_F=\frac{p_F^2(h)}{2m}+mgh$$

$$\rightarrow p_F(h) = \sqrt{2m\epsilon_F - 2m^2gh} = \sqrt{\left(3\pi^2 n_0\right)^{\frac{2}{3}} - 2m^2gh}$$

3. We assume that at zero height the fermions form a degenerate gas with Fermi energy ϵ_F^0, for $h > 0$ the energy form the momentum is

$$\epsilon_p = \epsilon_F^0 - mgh$$

The limit for non-degenerate fermions is $\epsilon_p(h_c) = T$

$$\epsilon_F^0 - mgh_c = T$$

$$\rightarrow h_c = \frac{\epsilon_F^0 - T}{mg} \approx \frac{\epsilon_F^0}{mg}$$

4. For $h \gg h_c$ the gas behave as classical gas, we have the same chemical equilibrium $\mu(h) = \mu(0)$, but now $\mu(0) = \epsilon_F^0 \approx mgh_c$

$$-T \ln\left(\frac{2}{n(h)\lambda_T^3}\right) + mgh = mgh_c$$

$$n(h) = \frac{2}{\lambda_T^3} e^{-\beta mg(h-h_c)}$$

For $h = 0$

$$n(0) = \frac{2}{\lambda_T^3} e^{\beta mgh_c}$$

$$\rightarrow n(h) = n_0 e^{-\beta mgh}$$