Statistical Mechanics - Class Exercise 9

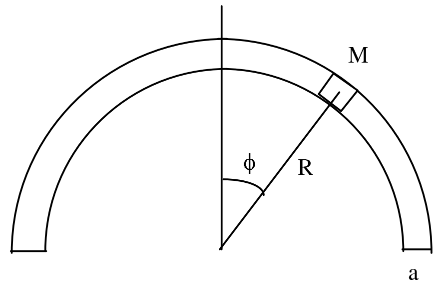
June 14, 2022

Exercise 5811 - Mechanical model for symmetry breaking

An airtight piston of mass M is free to move inside a cylindrical tube of cross sectional area a. The tube is bent into a semicircular shape of radius R. On each side of the piston there is an ideal gas of N atoms at a temperature T. The angular position of the piston is φ (see figure). The gravitation field of Earth exerts a force Mg on the piston, while its effect on the gas particles can be neglected.
The partition function of the system can be written as $d\varphi$ integral over $\exp[-A(\varphi)]$. The variable φ is regarded as the "order parameter" of the system. A small difference ΔN in the occupation of the two sides is regarded as the conjugate field. The susceptibility is defined via the relation $\langle\varphi\rangle \approx \chi\Delta N$.

1. Write an explicit expression for $A(\varphi)$.
2. Find the coefficients in the expansion $A(\varphi) = (a/2)\varphi^2 + (u/4)\varphi^4 - h\varphi$.
3. Deduce what is the critical temperature T_c.
4. Using Gaussian approximation find what is χ for $T > T_c$.
5. Using Gaussian approximation find what is χ for $T < T_c$.
6. Sketch a plot of χ versus T indicating by dashed lines the Gaussian approximations and by solid line the expected exact result. Write what is the range ΔT around T_c where the Gaussian approximation fails.
7. What is the way to take the "thermodynamic limit" such as to have a phase transition at finite temperature?
8. In reality, as the temperature is lowered, droplets condense on the walls of the left (larger) chamber. What do you expect to find in the right chamber (gas? liquid? both?).

Guidelines: In items (4) and (5) simplify the result assuming $T \sim T_c$ and express it in terms of T_c and $T - T_c$. The final answer should include one term only. Care about numerical prefactors - their correctness indicates that the algebra is done properly. In item (7) you are requested to identify the parameter that should be taken to infinity in order to get a "phase transition". Please specify what are the other parameters that should be kept constant while taking this limit.



Answer

1. The partition function of the system can be written as integral over φ

$$Z=\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}e^{-A(\varphi)}$$

For specific φ we have two gases and the piston

$$Z_{\varphi}=Z_{L}e^{-\beta MgR\cos(\varphi)}Z_{R}=e^{-A(\varphi)}$$

For the ideal gas in the two sides

$$Z_{L/R}=\frac{1}{N_{L/R}!}\left(\frac{aR\left(\frac{\pi}{2}\pm\varphi\right)}{\lambda_{T}^{3}}\right)^{N_{L/R}}$$

So we get

$$\begin{aligned}A\left(\varphi\right)=&-\ln\left(Z_{\varphi}\right)=\ln\left(N_{L}!\right)+\ln\left(N_{R}!\right)-N_{L}\ln\left(\frac{aR\left(\frac{\pi}{2}+\varphi\right)}{\lambda_{T}^{3}}\right)-N_{R}\ln\left(\frac{aR\left(\frac{\pi}{2}-\varphi\right)}{\lambda_{T}^{3}}\right)+\beta MgR\cos\left(\varphi\right)\\&=\ln\left(N_{L}!\right)+\ln\left(N_{R}!\right)-\left(N_{L}+N_{R}\right)\ln\left(\frac{aR\frac{\pi}{2}}{\lambda_{T}^{3}}\right)-N_{L}\ln\left(1+\frac{2\varphi}{\pi}\right)-N_{R}\ln\left(1-\frac{2\varphi}{\pi}\right)+\beta MgR\cos\left(\varphi\right)\end{aligned}$$

2. We assume $\varphi\ll1$ and a small difference ΔN in the occupation of the two sides

$$N_{L}=N+\frac{\Delta N}{2},\;\;N_{R}=N-\frac{\Delta N}{2}$$

So

$$A\left(\varphi\right)=\ln\left(\left(N+\frac{\Delta N}{2}\right)!\right)+\ln\left(\left(N-\frac{\Delta N}{2}\right)!\right)-2N\ln\left(\frac{aR\frac{\pi}{2}}{\lambda_{T}^{3}}\right)-N\ln\left(1-\left(\frac{2\varphi}{\pi}\right)^{2}\right)-\frac{\Delta N}{2}\left[\ln\left(1+\frac{2\varphi}{\pi}\right)-\ln\left(1-\frac{2\varphi}{\pi}\right)\right]+\beta$$

Now we use taylor expansion up to fourth order in φ, but for the part that multiple by ΔN ,that regarded as the external field, we take just first order

$$\begin{aligned}A\left(\varphi\right)\approx&const+N\left(\frac{2\varphi}{\pi}\right)^{2}+\frac{N}{2}\left(\frac{2\varphi}{\pi}\right)^{4}-\Delta N\frac{2\varphi}{\pi}+\frac{MgR}{T}\left(1-\frac{\varphi^{2}}{2}+\frac{\varphi^{4}}{24}\right)\\=&const+\frac{1}{2}\left(\frac{8N}{\pi^{2}}-\frac{MgR}{T}\right)\varphi^{2}+\frac{1}{4}\left(\frac{32N}{\pi^{4}}+\frac{MgR}{6T}\right)\varphi^{4}-\frac{2\Delta N}{\pi}\varphi\end{aligned}$$

So

$$a=\left(\frac{8N}{\pi^2}-\frac{MgR}{T}\right),\quad u=\left(\frac{32N}{\pi^4}+\frac{MgR}{6T}\right),\quad h=\frac{2\Delta N}{\pi}$$

3. The T_c is the temperature when $a\left(T_c\right)=0$,so

$$\frac{8N}{\pi^2}-\frac{MgR}{T_c}=0$$

$$T_c=\frac{\pi^2 MgR}{8N}$$

$$a=\frac{8N}{\pi^2}\left(1-\frac{\pi^2 MgR}{8NT}\right)=\frac{8N}{\pi^2}\left(\frac{T-T_c}{T}\right)$$

4. We need to find the minimum of $A\left(\varphi\right)$

$$A'\left(\varphi\right)=a\varphi+u\varphi^3-h=0$$

For $T>T_c$ we get $a\left(T\right)>0$, so for $h=0$ we have one solution

$$a\varphi+u\varphi^3=0\rightarrow\bar{\varphi}=0$$

So close to the minima we can neglect the fourth order and get

$$a\varphi-h=0\rightarrow\bar{\varphi}=\frac{h}{a}$$

And

$$A\left(\bar{\varphi}\right)\approx\frac{a}{2}\bar{\varphi}^2-h\bar{\varphi}=-\frac{h^2}{2a}$$

$$A\left(\varphi\right)\approx A\left(\bar{\varphi}\right)+\frac{1}{2}A''\left(\bar{\varphi}\right)\left(\varphi-\bar{\varphi}\right)^2=-\frac{h^2}{2a}+\frac{1}{2}a\left(\varphi-\bar{\varphi}\right)^2$$

So

$$Z=\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}e^{-A(\varphi)}\approx\int_{-\infty}^{\infty}e^{\frac{h^2}{2a}-\frac{1}{2}a(\varphi-\bar{\varphi})^2}=\sqrt{\frac{2\pi}{a}}e^{\frac{h^2}{2a}}$$

And

$$\left\langle\varphi\right\rangle=\frac{\partial\ln Z}{\partial h}=\frac{h}{a}=\frac{2}{a\pi}\Delta N$$

$$\chi=\frac{2}{a\pi}$$

For $T\approx T_c$

$$a=\frac{8N}{\pi^2}\left(\frac{T-T_c}{T}\right)\approx\frac{8}{\pi^2}N\left(\frac{T-T_c}{T_c}\right)$$

$$\chi=\frac{2}{a\pi}=\frac{\pi}{4N}\frac{T_c}{(T-T_c)}$$

5. For $T<T_c$ we get $a\left(T\right)<0$ and for $h=0$

$$a\varphi+u\varphi^3=0\rightarrow\bar{\varphi}=0,\pm\sqrt{\frac{|a|}{u}}$$

So for the two minima we get

$$A\left(\bar{\varphi}_{\pm}\right)=\frac{a}{2}\bar{\varphi}_{\pm}^{2}+\frac{u}{4}\bar{\varphi}_{\pm}^{4}-h\bar{\varphi}_{\pm}=-\frac{|a|^{2}}{2u}+\frac{|a|^{2}}{4u}\mp h\sqrt{\frac{|a|}{u}}=-\frac{|a|^{2}}{4u}\mp h\sqrt{\frac{|a|}{u}}$$

$$A''\left(\bar{\varphi}_{\pm}\right)=a+3u\bar{\varphi}_{\pm}^{2}=2\left|a\right|$$

$$A_{\pm}\left(\varphi\right)\approx A\left(\bar{\varphi}_{\pm}\right)+\frac{1}{2}A''\left(\bar{\varphi}_{\pm}\right)\left(\varphi-\bar{\varphi}_{\pm}\right)^{2}=-\frac{|a|^{2}}{4u}\mp h\sqrt{\frac{|a|}{u}}+|a|\left(\varphi-\bar{\varphi}_{\pm}\right)^{2}$$

We need to calculate around the two minima

$$Z=\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}e^{-A(\varphi)}\approx\int_{-\infty}^{\infty}e^{\frac{|a|^{2}}{4u}+h\sqrt{\frac{|a|}{u}}-|a|(\varphi-\bar{\varphi})^{2}}+\int_{-\infty}^{\infty}e^{\frac{|a|^{2}}{4u}-h\sqrt{\frac{|a|}{u}}-|a|(\varphi-\bar{\varphi})^{2}}$$

$$=\sqrt{\frac{\pi}{|a|}}e^{\frac{|a|^{2}}{4u}}\left(e^{h\sqrt{\frac{|a|}{u}}}+e^{-h\sqrt{\frac{|a|}{u}}}\right)=\sqrt{\frac{\pi}{|a|}}e^{\frac{|a|^{2}}{4u}}2\cosh\left(h\sqrt{\frac{|a|}{u}}\right)$$

$$\left\langle\varphi\right\rangle=\frac{\partial\ln Z}{\partial h}=\sqrt{\frac{|a|}{u}}\tanh\left(h\sqrt{\frac{|a|}{u}}\right)\approx\frac{|a|}{u}h=\frac{2}{\pi}\frac{|a|}{u}\Delta N$$

$$\chi=\frac{2}{\pi}\frac{|a|}{u}$$

For $T\approx T_c$

$$|a|=\frac{8}{\pi^{2}}N\left(\frac{T_{c}-T}{T_{c}}\right),\quad u=\left(\frac{32N}{\pi^{4}}+\frac{MgR}{6T}\right)=\frac{8N}{6\pi^{2}}\left(\frac{24}{\pi^{2}}+\frac{T_{c}}{T}\right)\approx\frac{4N}{3\pi^{4}}\left(24+\pi^{2}\right)$$

$$\chi=\frac{12\pi}{\left(24+\pi^{2}\right)}\left(\frac{T_{c}-T}{T_{c}}\right)$$

6. The condition for the Gaussian approximation is that the the quadratic term is dominant
For $T>T_c$we have $\bar{\varphi}=0$ and the condition is

$$a\varphi^{2}\gg u\varphi^{4}$$

$$\frac{a}{u}\gg\varphi^{2}$$

We can see that var$\varphi=\left\langle\left(\varphi-\bar{\varphi}\right)^{2}\right\rangle=\left\langle\varphi^{2}\right\rangle$, and in the other hand in the Gaussian approximation var$\varphi=\sigma^{2}=\frac{1}{a}$, so we get the condition

$$\frac{a}{u}\gg\frac{1}{a}$$

$$\frac{\left(\frac{8}{\pi^{2}}N\left(\frac{T-T_{c}}{T_{c}}\right)\right)^{2}}{\frac{4N}{3\pi^{4}}\left(24+\pi^{2}\right)}\gg1$$

$$\left(\frac{T-T_{c}}{T_{c}}\right)\gg\frac{1}{\sqrt{N}}$$

For $T<T_c$we have $\bar{\varphi}=\pm\sqrt{\frac{|a|}{u}}$ and the condition is

$$|a|\left(\varphi-\bar{\varphi}\right)^{2}\gg u\left(\varphi-\bar{\varphi}\right)^{4}$$

$$\frac{|a|}{u} \gg (\varphi - \bar{\varphi})^2 = \text{var}\varphi$$

in This limit $\sigma^2 = \frac{1}{2|a|}$, so we get

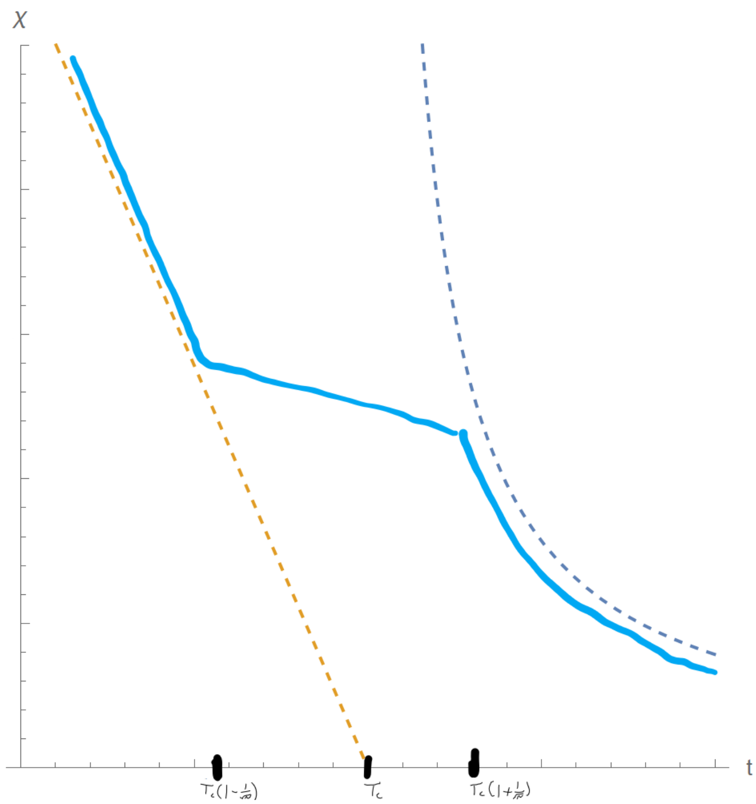
$$\frac{|a|}{u} \gg \frac{1}{2\,|a|}$$

$$\left(\frac{T_c - T}{T_c}\right) \gg \frac{1}{\sqrt{N}}$$

so we can conclude

$$\left|\frac{T - T_c}{T_c}\right| \gg \frac{1}{\sqrt{N}}$$

$$T \gg T_c\left(1 + \frac{1}{\sqrt{N}}\right) \;\; \text{OR} \;\; T \ll T_c\left(1 - \frac{1}{\sqrt{N}}\right)$$



7. The “thermodynamic limit” is when $N \to \infty$, but the density $n = \frac{N}{a\pi R}$ stay constant. if we want to get a phase transition at finite temperature we need to keep $T_c = \frac{\pi^2 MgR}{8N}$ as a constant,so we need to keep $\frac{R}{N} = const$. in this condition we get a phase transition

$$\chi = \begin{cases} \frac{12\pi}{(24+\pi^2)}\left(\frac{T_c - T}{T_c}\right) & T < T_c \\ 0 & T > T_c \end{cases}$$

8. Assuming the system is in equilibrium we have $T_L = T_R$ and $P_R = P_L + \frac{Mg|sin(\varphi)|}{a}$. Remember that the state of coexistence of liquid and gas is a line $P(T)$, assuming the state of the side with the larger volume is on the line, since $P_R > P_L$ the state of the side with the smaller volume must be above the line thus contains liquid only.