Statistical Mechanics - Class Exercise 11

June 27, 2022

Exercise 8481 - Mass on a spring

A balance for measuring weight consists of a sensitive spring which hangs from a fixed point. The spring constant is K. The balance is at temperature T and gravity acceleration is g in the x direction. A small mass m hangs at the end of the spring. There is an option to apply an external force $F(t)$, to which x is conjugate or apply an external vector potential $A(t)$.

1. Find the partition function Z.
2. Find $\langle x \rangle$ and $\langle x^2 \rangle$ and $\mathrm{Var}(x)$.
3. Write a Langevin equation for $x(t)$, with friction η, and a random force $f(t)$.
4. Assuming $\langle f(t)f(0) \rangle = C\delta(t)$, find $\mathrm{Var}(x)$, and deduce what is C by comparing with the canonical result.
5. Assuming x is measured in the lab by averaging over time period t_0, what is the minimal mass that can be meaningfully measured?
6. Describe the external force $F(t)$ by a scalar potential and demonstrate FDT.
7. Describe the external force $F(t)$ by a vector potential and demonstrate FDT.

Note: $\int \frac{d\omega}{(\omega^2-\omega_0^2)^2+\gamma^2\omega^2} = \frac{\pi}{\gamma\omega_0^2}$.

Answer

1. The Hamiltonian of the system is:

$$H = \frac{p^2}{2m} + \frac{1}{2}Kx^2 - mgx$$

We can rewrite the Hamiltonian in the following way

$$\begin{aligned} H =& \frac{p^2}{2m} + \frac{1}{2}K\left(x^2 - 2\frac{mg}{K}x + \left(\frac{mg}{K}\right)^2\right) - \frac{(mg)^2}{2K} \\ =& \frac{p^2}{2m} + \frac{1}{2}K\left(x - x_0\right)^2 - \frac{(mg)^2}{2K}, x_0 = \frac{mg}{K} \end{aligned}$$

So, the partition function

$$\frac{e^{\frac{\beta(mg)^2}{2K}}}{\lambda_T}\int_{-\infty}^{\infty} e^{-\beta\frac{K}{2}(x-x_0)^2}dx = \frac{e^{\frac{\beta(mg)^2}{2K}}}{\lambda_T}\sqrt{\frac{2\pi}{\beta K}}$$

2. The Gaussian in the partition function is centered around x_0, therefore we deduce that

$$\langle x\rangle = \frac{1}{Z_x}\int_{-\infty}^{\infty} x e^{-\beta\frac{K}{2}(x-x_0)^2}dx = \frac{1}{Z_x}\int_{-\infty}^{\infty}(x+x_0)\,e^{-\beta\frac{K}{2}x^2}dx = x_0$$

In order to find $\langle x^2\rangle$ we use the equipartition and the Virial theorems

$$\langle x\cdot\frac{\partial U}{\partial x}\rangle = \langle p\cdot\frac{\partial \mathcal{K}}{\partial p}\rangle = \left\langle\frac{p^2}{m}\right\rangle = T$$

$$\langle xK(x-x_0)\rangle = T$$

$$\langle x^2\rangle = \frac{T}{K}+x_0^2$$

And the variance of x is

$$\text{Var}(x) = \langle x^2\rangle - \langle x\rangle^2 = \frac{T}{K}$$

3. The Langevin equation for this system is:

$$\dot{x} = \frac{\partial H}{\partial p}, \dot{p} = -\frac{\partial H}{\partial x}$$

$$m\ddot{x}+\eta\dot{x}+Kx-mg = f(t)$$

4. First we change the variable $x\to x-x_0$, that in equilibrium $\langle x\rangle = 0$,the Langevin equation became

$$m\ddot{x}+\eta\dot{x}+Kx = f(t)$$

After Fourier transform

$$(-m\omega^2 - i\eta\omega + K)x_\omega = f_\omega$$

Multiply by the conjugate the both sides and averege

$$\left(\left(K-m\omega^2\right)^2+\eta^2\omega^2\right)\left\langle|x_\omega|^2\right\rangle = \left\langle|f_\omega|^2\right\rangle$$

Using the Wiener-Khinchin theorem $\left\langle|f_\omega|^2\right\rangle = \tilde{C}_{ff}(\omega)\times t$, we get

$$\tilde{C}_{xx}(\omega) = \frac{1}{m^2}\frac{\tilde{C}_{ff}(\omega)}{(\omega^2-\frac{K}{m})^2+(\gamma\omega)^2}, \gamma = \frac{\eta}{m}$$

The Fourier transform of $\delta(t)$ is 1, so the force-correlation is $\tilde{C}_{ff}(\omega) = C$. We get

$$\text{Var}(x) = C_{xx}(t=0) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\tilde{C}_{xx}(\omega)d\omega = \frac{1}{2\pi}\frac{C}{m^2}\int_{-\infty}^{\infty}\frac{d\omega}{(\omega^2-\frac{K}{m})^2+(\gamma\omega)^2} = \frac{C}{2\eta K}$$

when we use $\int\frac{d\omega}{(\omega^2-\omega_0^2)^2+\gamma^2\omega^2} = \frac{\pi}{\gamma\omega_0^2}$
so we get

$$\text{Var}(x) = \frac{C}{2\eta K} = \frac{T}{K}$$

$$C = 2\eta T = \nu$$

Finally we get

$$\tilde{C}_{xx}(\omega) = \frac{T}{m}\frac{2\gamma}{(\omega^2 - \frac{K}{m})^2 + (\gamma\omega)^2}$$

5. We measure $x(t)$ in the lab and average over time period t_0. This measurement introduces a new random variable X

$$X = \frac{1}{t_0}\int_0^{t_0} x(t)dt$$

This variable has a mean value $\langle X\rangle$ and variance $\text{Var}(X)$. In order to get a meaningful measurement, it has to obey the condition - $\langle X\rangle \gg \sqrt{\text{Var}(X)}$. From this condition, the minimal mass m_{min} will be found. The mean value of X is

$$\langle X\rangle = \frac{1}{t_0}\int_0^{t_0} \langle x(t)\rangle\, dt = x_0$$

The variance (after we change variables $x \to x - x_0$)

$$\text{Var}(X) = \frac{1}{t_0^2}\int_0^{t_0} dt' \int_0^{t_0} dt'' \langle x(t')x(t'')\rangle = \frac{1}{t_0}\tilde{C}_{xx}(\omega = 0) = \frac{2\eta T}{t_0 K^2}$$

so, the minimal mass is given by

$$x_0 = \frac{mg}{K} \gg \sqrt{\frac{2\eta T}{t_0 K^2}}$$

$$m \gg \sqrt{\frac{2\eta T}{g^2 t_0}} = m_{\text{min}}$$

6. The force $F(t)$ is described by a scalar potential U so the interaction term is $-\varepsilon(t)\,x$, so the conjugate variables are x and ε.
Averaging the Langevin formula

$$m\langle\ddot{x}\rangle + \eta\langle\dot{x}\rangle + K\langle x\rangle = \varepsilon$$

After Fourier transforming

$$x_\omega = \frac{1}{(-m\omega^2 + K) - i\omega\eta}\epsilon_\omega = \chi_\omega\varepsilon_\omega$$

Hence, we get the correlation function from FDT

$$\tilde{C}_{xx}(\omega) = \frac{2T}{\omega}\text{Im}\chi_\omega = \frac{T}{m}\frac{2\gamma}{(\omega^2 - \frac{K}{m})^2 + (\gamma\omega)^2}$$

7. The force $F(t)$ is described by a vector potential $A(t)$, the interaction term is $v \cdot A$, so the conjugate variables are v and $-A$. The averaged Langevin formula becomes

$$m\langle\dot{v}\rangle + \eta\langle v\rangle + K\langle x\rangle = -\dot{A}$$

After Fourier transforming

$$v_\omega = \frac{1}{\frac{i}{\omega}(-m\omega^2 + K) + \eta} i\omega A_\omega$$

$$v_\omega = \frac{1}{m}\frac{\omega^2}{(-\omega^2 + \frac{K}{m}) - i\omega\gamma}A_\omega$$

We use that $F(t) = -\dot{A} \to F_\omega = i\omega A_\omega$
The correlation function is

$$v_\omega = \chi_\omega A_\omega$$

$$\tilde{C}_{vv}(\omega) = \frac{2T}{\omega}\mathrm{Im}\chi_\omega = \frac{T}{m}\frac{2\gamma\omega^2}{(\omega^2 - \frac{K}{m})^2 + (\gamma\omega)^2} = \omega^2\tilde{C}_{xx}(\omega)$$

what we can get immediately from

$$\tilde{C}_{vv}(\omega) = \frac{1}{t}\left\langle |v_\omega|^2 \right\rangle = \frac{1}{t}\omega^2\left\langle |x_\omega|^2 \right\rangle = \omega^2\tilde{C}_{xx}(\omega)$$

Exercise 8483 - Millikan experiment

Consider a Millikan-type experiment whose purpose is to measure the charge e of a particle with mass m. The particle is located between plates of capacitor, where the electric field $\mathcal{E}$ is in the "up" direction, while the gravitation g is in the "down" direction. The distance between the plates is L, and the temperature of the system is T. Due to the poor vacuum the particle executes a Brownian motion that is described by a Langevin equation with friction force $-\eta v$. The charge of the electron is estimated via $\delta F = e\mathcal{E} - mg = 0$. In item (1) the system is prepared with a single particle in the middle. In item (3) assume a uniform gas of N particles. In both cases the current is integrated during a time interval t, and the charge $Q = \int I(t')dt'$ is inspected as "readout".

1. Assuming that $\delta F = 0$, determine the time t_d such that for $t \ll t_d$ it is not likely to get charge readout.
2. What is the δF for which the condition $t \ll t_d$ is no longer valid. We shall regard this value, call it δ_1, as the resolution of the measurement.
3. Assuming that $\delta F = 0$, determine the power spectrum $C(\omega)$ of the current $I(t)$.
4. Assume that the time of the measurement is t. What is the δF for which the condition $\langle Q\rangle \gg \sqrt{\mathrm{var}(Q)}$ is no longer valid. We shall regard this value, call it δ_N, as the resolution of the measurement.
5. Express the ratio δ_N/δ_1 as a function of N and t/t_d.

Tips: In the absence of fluctuations $\delta F = 0$ is indicated by having zero readout. In item (3) the "readout" is a current versus voltage ("IV") measurement, and $\delta F = 0$ is indicated by zero current. Due to the fluctuations there is some blurring which determines the resolution δ_N. In order to calculate the fluctuations in item (3) define the one-particle current as the velocity (up to a prefactor).

Answer

1. The Langevin equation for the Brownian motion:

$$m\dot{v} + \eta v = f(t)$$

with $\langle f(t)\rangle = 0$ so for steady state $\langle v\rangle = 0$.
Solving this equation for the spreading of the particle yields $\left\langle (x(0) - x(t))^2 \right\rangle = 2Dt$ where $D = \frac{T}{\eta}$. It follows, that it would be unlikely to get a charge readout for:

$$\frac{T}{\eta}\cdot t \ll L^2 \longrightarrow t_d = \frac{\eta L^2}{T}$$

2.When $e\mathcal{E} - mg \neq 0$ a “drift” term is to be added to the Langevin equation:

$$m\dot{v} + \eta v = f(t) + \delta F$$

So now the average velocity is $\langle v\rangle = \frac{\delta F}{\eta}$. In this case a minimum measurment time is $t = \frac{L}{\langle v\rangle}$. But we would also want this time to be shorter than the spreading time t_d we found in the previous item. This leads to the condition:

$$\frac{L}{\langle v\rangle} < t < \frac{\eta L^2}{T}$$

$$\delta F > \frac{T}{L} \equiv \delta_1$$

3. The current of a single particle is $I^1 = \frac{e}{L}v$. The power spectrum can be expressed as:

$$\left\langle |I^1_\omega|^2 \right\rangle = \left(\frac{e}{L}\right)^2 \left\langle |v_\omega|^2 \right\rangle$$

For Wiener-Khinchin theorem

$$C_{II}\left(\omega\right) = \left(\frac{e}{L}\right)^2 C_{vv}\left(\omega\right)$$

The Langevin equation:

$$\left(\eta - i\omega m\right) v_\omega = f_\omega$$

$$\left\langle |v_\omega|^2 \right\rangle = \frac{1}{\eta^2 + m^2\omega^2} \left\langle |f_\omega|^2 \right\rangle$$

$$C_{vv}\left(\omega\right) = \frac{C_{ff}\left(\omega\right)}{\eta^2 + m^2\omega^2}$$

when, for white noise $C_{ff}\left(\omega\right) = \nu = 2\eta T$, so

$$C_{II}\left(\omega\right) = \left(\frac{e}{L}\right)^2 \frac{T}{m}\frac{2\gamma}{\gamma^2 + \omega^2}, \gamma = \frac{\eta}{m}$$

The total current is a sum over single particle currents and so the power of the total current will be N times the power from a single particle:

$$C_{II}\left(\omega\right) = N\left(\frac{e}{L}\right)^2 \frac{T}{m}\frac{2\gamma}{\omega^2 + \gamma^2}$$

4. The readout is the total charge $Q = \int_0^t I(t')dt'$. For a significant readout we require $\sqrt{\mathrm{var}(Q)} \ll \langle Q\rangle$.

$$\langle Q\rangle = \langle I\rangle t = N\frac{e}{L}\langle v\rangle t = N\frac{e}{L}\frac{\delta F}{\eta}t$$

in the other side (that calculated assuming $\delta F = 0$):

$$\text{Var}(Q) = \langle Q^2 \rangle = \int_0^t \int_0^t dt'dt'' \langle I(t')I(t'') \rangle = C_{II}(\omega = 0)\, t = N \left(\frac{e}{L}\right)^2 \frac{2T}{\eta} t$$

The condition on δF is then:

$$\sqrt{N \left(\frac{e}{L}\right)^2 \frac{2T}{\eta} t} \ll N \frac{e}{L} \frac{\delta F}{\eta} t$$

$$\delta F > \sqrt{\frac{T\eta}{Nt}} \equiv \delta_N$$

5. The ratio $\frac{\delta_N}{\delta_1}$ can be expressed as a function of N and t/t_d:

$$\frac{\delta_N}{\delta_1} = \frac{1}{\sqrt{N\frac{t}{t_d}}}$$

Exercise 8490 - Stochastic rate equation

Consider N classical particles in a two site system. The two sites are subjected to a potential difference ε. The temperature of the system is T. Define $n \in [-N, N]$ as the occupation difference. In items (3-6) assume that the thermalization process can be described by a stochastic rate equation

$$\frac{dn}{dt} = -\gamma n + A(t)$$

where $A(t)$ is a noisy term that reflects the fluctuations of the potential difference. Assuming that it has an average value A_0 and a power spectrum $\phi(\omega)$, it follows that n relaxes to an average value $\langle n \rangle$, with fluctuations that are characterized by a power spectrum $C(\omega)$.

1. Write what is the interaction energy H_{int} of n with the field ε. Later you will have to be careful with the identification of the conjugate variables.
2. Using the canonical formalism find what are $\langle n \rangle$ and $\text{Var}(n)$. Additionally provide approximations for small ε.
3. Determined what is A_0 such that $\langle n \rangle$ would be consistent with the canonical result. Assuming small ε deduce that $A_0 \propto \varepsilon$, and find the pre-factor.
4. What is the $\chi(\omega)$ that characterizes the response of n to the applied potential in the linear-response regime? Assume that the dynamics is described by the stochastic rate equation; care to identify correctly the conjugate variables; and take into account your answer to item (3).
5. Deduce from the fluctuation-dissipation relation what is the power spectrum $C(\omega)$. Care to use the appropriate definition for $\chi(\omega)$, else the result will come out wrong.
6. Deduce what is the power spectrum $\phi(\omega)$ that is required in order to reproduce $C(\omega)$ from the stochastic rate equation.

Advice: In item (5) verify that your result is consistent with the answer to item (2). Likewise you can debug the numerical pre-factor in your answer to item (6). Care about factors of "2" in your answers. Failure to provide strictly correct pre-factors will be regarded as an essential error.

Answer

1. We take the potential difference ε in That way, potential of $-\frac{\varepsilon}{2}$ in site 1and $\frac{\varepsilon}{2}$ in site 2

$$\mathcal{H}_{\text{int}} = -\frac{\varepsilon}{2}N_1 + \frac{\varepsilon}{2}N_2 = -\frac{\varepsilon}{2}n$$

2. The partition function is

$$Z_1 = e^{\beta\frac{\varepsilon}{2}} + e^{-\beta\frac{\varepsilon}{2}} = 2\cosh\left(\beta\frac{\varepsilon}{2}\right)$$

$$Z = (Z_1)^N = 2^N \cosh^N\left(\beta\frac{\varepsilon}{2}\right)$$

From this we get

$$\langle n\rangle = \frac{1}{\beta}\frac{\partial \ln(Z)}{\partial\frac{\varepsilon}{2}} = N\tanh\left(\beta\frac{\varepsilon}{2}\right)$$

We notice that in the limit $\varepsilon \to 0$ we have $\langle n\rangle \to 0$ and in the limit $\varepsilon \to \infty$ we have $\langle n\rangle \to N$, as expected.

$$\text{Var}(n) = \frac{1}{\beta^2}\frac{\partial^2 \ln(Z)}{\partial\frac{\varepsilon}{2}^2} = \frac{N}{\cosh^2\left(\beta\frac{\varepsilon}{2}\right)}$$

If we approximate for small ε we get

$$\langle n\rangle \approx \frac{N}{T}\frac{\varepsilon}{2}$$

$$\text{Var}(n) \approx N\left(1 - \left(\frac{1}{T}\frac{\varepsilon}{2}\right)^2\right)$$

3. The stochastic rate equation

$$\frac{dn}{dt} = -\gamma n + A(t)$$

after averaging we get in steady state

$$\langle n\rangle = \frac{A_0}{\gamma}$$

we require

$$\frac{A_0}{\gamma} = \frac{N}{T}\frac{\varepsilon}{2}$$

$$\to A_0 = \gamma\frac{N}{T}\frac{\varepsilon}{2}$$

4. By taking the Fourier transform of the averaging stochastic rate equation

$$n_\omega = \frac{A_{0\omega}}{(\gamma - i\omega)}$$

$$n_\omega = \frac{\gamma N\beta}{(\gamma - i\omega)}\frac{\varepsilon_\omega}{2}$$

$$\to \chi(\omega) = \frac{\gamma N\beta}{(\gamma - i\omega)}$$

5.

$$\chi(\omega)=\frac{\gamma N\beta}{(\gamma-i\omega)}=\frac{\gamma^2 N\beta}{(\gamma^2+\omega^2)}+i\frac{\gamma N\beta\omega}{(\gamma^2+\omega^2)}$$

From FDT we get

$$\mathrm{Im}\chi(\omega)=\tanh\left(\frac{\omega}{2T}\right)C_{nn}(\omega)$$

in the classical limit $\omega\to 0$

$$\frac{\mathrm{Im}\chi(\omega)}{\omega}=\frac{1}{2T}C_{nn}(\omega)=\frac{\gamma N\beta}{\gamma^2+\omega^2}$$

$$C_{nn}(\omega)=N\frac{2\gamma}{\gamma^2+\omega^2}$$

6.

$$\left\langle |n_\omega|^2\right\rangle=\frac{\left\langle |A_\omega|^2\right\rangle}{\gamma^2+\omega^2}$$

$$C_{nn}(\omega)=\frac{\phi(\omega)}{\gamma^2+\omega^2}$$

$$\phi(\omega)=N2\gamma$$

Exercise 8034 - Brownian particle on a ring

The motion of a classical Brownian particle on a 1D ring is described by the Langevin equation $m\ddot{\theta}+\eta\dot{\theta}=f(t)$, where $f(t)$ is due to a noisy electromotive force that has a correlation function $\langle f(t')f(t'')\rangle=C_f(t'-t'')$. The power spectrum $\tilde{C}_f(\omega)$ is defined as the Fourier transform of the correlation function. We consider two cases:

1. High temperature white noise $\tilde{C}_f(\omega)=\nu$.
2. Zero temperature noise $\tilde{C}_f(\omega)=c|\omega|$.

We define the angular velocity of the particle as $v=\dot{\theta}$, and its Cartesian coordinate as $x=\sin(\theta)$. In the absence of noise the dynamics is characterized by the damping time $t_c=m/\eta$.
In items (3)-(5) you should assume a spreading scenario: the particle is initially ($t=0$) located at $\theta\sim 0$. The spreading during the transient period $0<t<t_c$ is assumed to be negligible. In item (6) assume that the particle had been launched in the far past ($t=-\infty$): accordingly there is no preferred location on the ring.

1. Find the exact correlation function $\langle v(t)v(0)\rangle$ in case (a).
2. Find the correlation function $\langle v(t)v(0)\rangle$ for $t\gg t_c$ in case (b).
3. Find the spreading $S(t)\equiv\langle\theta(t)^2\rangle$ for $t\gg t_c$ in case (a).
4. Find the spreading $S(t)\equiv\langle\theta(t)^2\rangle$ for $t\gg t_c$ in case (b).
5. Express $\langle x(t)^2\rangle$ for a spreading scenario given $S(t)$.
6. Express the correlation function $\langle x(t)x(0)\rangle$ given $S(t)$.

7. Write the explicit long time expression for $\langle x(t)x(0)\rangle$ in case (b), and deduce what is the critical value η_c above which a "phase transition" is expected in the response characteristics of the system.

Tips: For a Gaussian variable that has zero average $\langle \exp i\varphi \rangle = \exp[-(1/2)\langle \varphi^2 \rangle]$.
The Fourier transform of $|\omega|$ has zero area, with negative tails $-1/(\pi t^2)$.
If you fail to solve (6), assume that the answer is the same as in (5), and proceed to (7).

Answer

1. We will start with writing the Langevin equation for the velocity $m\dot{v} + \eta v = f(t)$, we can solve it with Fourier transform:

$$(-i\omega m + \eta)v_\omega = f(\omega)$$

$$v_\omega = \frac{1}{m}\frac{f(\omega)}{\gamma - i\omega}, \gamma = \frac{\eta}{m}$$

Now we can take square absolute value from both sides and average :

$$\langle |v_\omega|^2 \rangle = \frac{1}{m^2}\frac{\langle |f(\omega)|^2 \rangle}{\gamma^2 + \omega^2}$$

From the Wiener-Khinchin theorem we get that $\langle |f(\omega)|^2 \rangle = \tilde{C}_f(\omega) \times t$, so we get:

$$\tilde{C}_v(\omega) = \frac{1}{m^2}\frac{\tilde{C}_f(\omega)}{\gamma^2 + \omega^2}$$

For case (a) $\tilde{C}_f(\omega) = \nu$ we get:

$$\tilde{C}_v(\omega) = \frac{1}{m^2}\frac{\nu}{\gamma^2 + \omega^2}$$

After inverse Fourier transform:

$$C_v(t) = \int \frac{d\omega'}{2\pi}\frac{1}{m^2}\frac{\nu}{\gamma^2 + \omega'^2}e^{-i\omega' t} = \frac{\nu}{2m^2\gamma}\int d\omega \frac{1}{\pi}\frac{\gamma}{\gamma^2 + \omega'^2}e^{-i\omega' t}$$

This is a Lorentzian, so we get:

$$C_v(t) = \langle v(t)v(0)\rangle = \frac{\nu}{2m^2\gamma}e^{-\gamma|t|} = \frac{\nu}{2m^2\gamma}e^{-\frac{|t|}{t_c}}$$

2. Now we use the same equation but in case (b) $\tilde{C}_f(\omega) = c|\omega|$:

$$\tilde{C}_v(\omega) = \frac{1}{m^2}\frac{\tilde{C}_f(\omega)}{\gamma^2 + \omega^2} = \frac{1}{m^2}\frac{c|\omega|}{\gamma^2 + \omega^2}$$

we need to do inverse Fourier transform:

$$C_v(t) = \frac{c}{m^2}\int \frac{d\omega'}{2\pi}\frac{|\omega'|}{\gamma^2 + \omega'^2}e^{-i\omega' t}$$

We know that the change of $|\omega|$ is slow except near the $\omega = 0$, so the shape of $\omega \approx 0$ is determined by the higher t and the shape of $\omega \gg 0$ is determined by the lower t. We take the limit $t \gg t_c$ so we can neglect $\omega > \frac{1}{t_c}$ and get:

$$C_v(t) = \langle v(t)v(0)\rangle = \frac{c}{m^2}\int \frac{d\omega'}{2\pi}\frac{|\omega'|}{\gamma^2+\omega'^2}e^{-i\omega' t} = \frac{c}{\eta^2}\int \frac{d\omega'}{2\pi}\frac{|\omega'|}{1+t_c^2\omega'^2}e^{-i\omega' t} \approx \frac{c}{\eta^2}\int \frac{d\omega'}{2\pi}|\omega'|e^{-i\omega' t}$$

We know that the Fourier transform of $|\omega|$ has zero area, with negative tails $-\frac{1}{\pi t^2}$, so we get:

$$\begin{aligned}\int \frac{d\omega'}{2\pi}|\omega'|e^{-i\omega' t} &= -\int_{-\infty}^{0}\frac{d\omega'}{2\pi}\omega' e^{-i\omega' t} + \int_{0}^{\infty}\frac{d\omega'}{2\pi}\omega' e^{-i\omega' t}\\ &= i\partial_t \lim_{\eta\to 0}\int_{-\infty}^{0}\frac{d\omega'}{2\pi}e^{-i\omega' t+\eta\omega'} + i\partial_t \lim_{\eta\to 0}\int_{0}^{\infty}\frac{d\omega'}{2\pi}e^{-i\omega' t-\eta\omega'}\\ &= -i\partial_t \lim_{\eta\to 0}\frac{1}{2\pi}\frac{1}{-it+\eta} - i\partial_t \lim_{\eta\to 0}\frac{1}{2\pi}\frac{1}{-it-\eta} = \frac{1}{\pi}\partial_t\frac{1}{t} = -\frac{1}{\pi t^2}\end{aligned}$$

$$C_v(t) = \langle v(t)v(0)\rangle \approx \frac{c}{\eta^2}\int \frac{d\omega'}{2\pi}|\omega'|e^{-i\omega' t} = -\frac{c}{\eta^2\pi t^2}$$

3. In the beginning we define $\dot{\theta} = v$, so we get:

$$\theta(t) = \int_0^t dt' v(t')$$

$$\theta^2(t) = \int_0^t\int_0^t dt'dt'' v(t')v(t'')$$

$$\langle\theta^2(t)\rangle = \int_0^t\int_0^t dt'dt''\langle v(t')v(t'')\rangle = \int_0^t\int_0^t dt'dt'' C_v(t'-t'')$$

We can see that t', t'' are independent variables, so we can choose that $t' > t''$ and double the result. We can do a change of variables to two dependent variables $T = t' \to 0 < T < t, \tau = t' - t'' \to 0 < \tau < T$.

$$\langle\theta^2(t)\rangle = 2\int_0^t dT\int_0^T d\tau C_v(\tau) = 2\int_0^t dT\int_0^T d\tau \frac{\nu}{2m\eta}e^{-\frac{|\tau|}{\left(\frac{m}{\eta}\right)}}$$

The correlation decay very fast so in the limit $t \gg t_c$ we can take the integral to infinite:

$$\langle\theta^2(t)\rangle = 2\int_0^t dT\int_0^\infty d\tau \frac{\nu}{2m\eta}e^{-\frac{|\tau|}{\left(\frac{m}{\eta}\right)}} = 2t\frac{\nu}{2m\eta}\left(\frac{m}{\eta}\right) = \frac{\nu}{\eta^2}t$$

Or, in the short way

$$\langle\theta^2(t)\rangle = \int_0^t\int_0^t dt'dt''\langle v(t')v(t'')\rangle = \int_0^t\int_0^t dt'dt'' C_v(t'-t'') = \tilde{C}_v(\omega=0)\cdot t = \frac{\nu}{\eta^2}t$$

4. In the same way:

$$\langle\theta^2(t)\rangle = 2\int_0^t dT\int_0^T d\tau C_v(\tau)$$

We can neglect the spreading during the transient period $0 < t < t_c$, so we take the limit:

$$\langle\theta^2(t)\rangle = 2\int_{t_c}^{t} dT \int_0^T d\tau C_v(\tau)$$

The solution we found to $C_v(t)$ in case (b) is good just for $t \gg t_c$, so we need to divide the integral to two parts (we assume that the limit $T = t_c$ is the lower limit to our solution):

$$\langle\theta^2(t)\rangle = 2\int_{t_c}^{t} dT \left(\int_0^\infty d\tau C_v(\tau) - \int_T^\infty d\tau C_v(\tau)\right)$$

The part $\int_0^\infty d\tau C_v(\tau) = \tilde{C}_v(\omega = 0) = 0$, so we get:

$$\langle\theta^2(t)\rangle = -2\int_{t_c}^{t} dT \int_T^\infty d\tau C_v(\tau) = 2\int_{t_c}^{t} dT \int_T^\infty d\tau \frac{c}{\eta^2\pi\tau^2} = 2\int_{t_c}^{t} dT \frac{c}{\eta^2\pi T} = \frac{2c}{\pi\eta^2}\ln\frac{|t|}{t_c}$$

5. We defined $x = \sin\theta$:

$$\langle x^2(t)\rangle = \langle\sin^2\theta(t)\rangle = \langle\frac{\left(e^{i\theta} - e^{-i\theta}\right)^2}{-4}\rangle = \frac{1}{4}\langle\left(2 - e^{i2\theta} - e^{-i2\theta}\right)\rangle = \frac{1}{2} - \frac{1}{4}\langle e^{i2\theta}\rangle - \frac{1}{4}\langle e^{-i2\theta}\rangle$$

We get a tip that for a Gaussian variable that has zero average $\langle e^{i\varphi}\rangle = e^{-(1/2)\langle\varphi^2\rangle}$. Because θ is a Gaussian variable and it has zero average we get:

$$\langle x^2(t)\rangle = \frac{1}{2} - \frac{1}{4}\langle e^{i2\theta}\rangle - \frac{1}{4}\langle e^{-i2\theta}\rangle = \frac{1}{2} - \frac{1}{4}e^{-2\langle\theta^2\rangle -} - \frac{1}{4}e^{-2\langle\theta^2\rangle} = \frac{1}{2}\left(1 - e^{-2\langle\theta^2\rangle}\right) = \frac{1}{2}\left(1 - e^{-2S(t)}\right)$$

Note that $\langle(2\theta)^2\rangle = \langle(-2\theta)^2\rangle = 4\langle\theta^2\rangle$

6. In the previous sections we assumed that $\theta(0) \approx 0$ and we talked about short times, so we could treat θ like a coordinate and calculate $\langle\theta(t)^2\rangle$. Now there isn't a preferred location on the ring so we can't calculate $S(t) = \langle\theta(t)^2\rangle$, because θ is not well defined. So we can't calculate $\langle x^2(t)\rangle$ like before, just the correlation between two different times $\langle x(t)x(0)\rangle$.

By definition:

$$\langle x(t)x(0)\rangle = \int_0^{2\pi}\int_0^{2\pi} \sin(\theta_t)\sin(\theta_0)\rho(\theta_t,\theta_0)d\theta_t d\theta_0$$

The formula for conditional probability is:

$$\rho(A|B) = \frac{\rho(A,B)}{\rho(B)} \to \rho(A,B) = \rho(A|B)\rho(B)$$

$$\langle x(t)x(0)\rangle = \int_0^{2\pi}\int_0^{2\pi} \sin(\theta_t)\sin(\theta_0)\rho(\theta_t|\theta_0)\rho(\theta_0)d\theta_t d\theta_0$$

We get that in $t = 0$, θ_0 has a Uniform distribution, namely $\rho(\theta_0) = \frac{1}{2\pi}$.

Additionally $\rho(\theta_t|\theta_0)$ is the probability to find θ_t when we know where is θ_0, and this is like the previous section, when we assumed that $\theta_0 = 0$.

The probability $\rho(\theta_t|\theta_0)$ depends only on the difference between θ_t and θ_0, it doesn't depends on one of them, so let's define $\delta\theta = \theta_t - \theta_0$, when $\rho(\theta_t|\theta_0)d\theta_t = \rho(\delta\theta)d\delta\theta$

By using a trigonometric identities we get:

$$\int_0^{2\pi}\int_0^{2\pi}\sin(\theta_t)\sin(\theta_0)\rho(\theta_t|\theta_0)\rho(\theta_0)d\theta_t d\theta_0 = \frac{1}{4\pi}\int_0^{2\pi}\int_0^{2\pi}\left(\cos(\delta\theta)-\cos(2\theta_0+\delta\theta)\right)\rho(\delta\theta)d\delta\theta d\theta_0 =$$

$$= \frac{1}{4\pi}\int_0^{2\pi}\int_0^{2\pi}\cos(\delta\theta)\rho(\delta\theta)d\delta\theta d\theta_0 - \frac{1}{4\pi}\int_0^{2\pi}\int_0^{2\pi}\cos(2\theta_0+\delta\theta)\rho(\delta\theta)d\delta\theta d\theta_0 =$$

The first integral doesn't depend on θ_0, the integral on θ_0 in the second term give 0, and we get:

$$= \frac{1}{4\pi}\int_0^{2\pi}\int_0^{2\pi}\cos(\delta\theta)\rho(\delta\theta)d\delta\theta d\theta_0 = \frac{1}{2}\int_0^{2\pi}\cos(\delta\theta)\rho(\delta\theta)d\delta\theta = \frac{1}{2}\langle\cos(\delta\theta)\rangle$$

When we take $\theta_0 = 0$ we get that $\delta\theta$ is the same θ we define in the previous section and get:

$$\frac{1}{2}\langle\cos(\delta\theta)\rangle = \frac{1}{4}\langle e^{i\delta\theta}+e^{-i\delta\theta}\rangle = \frac{1}{4}\left(\langle e^{i\delta\theta}\rangle+\langle e^{-i\delta\theta}\rangle\right) = \frac{1}{2}e^{-\frac{1}{2}\langle\delta\theta^2\rangle} = \frac{1}{2}e^{-\frac{1}{2}S(t)}$$

7. For case (b):

$$S(t) = \frac{2c}{\pi\eta^2}\ln\frac{t}{t_c}$$

$$\langle x(t)x(0)\rangle = \frac{1}{2}e^{-\frac{1}{2}S(t)} = \frac{1}{2}\left(\frac{t}{t_c}\right)^{-\frac{c}{\pi\eta^2}}$$

From the FDT we get the relationship between the correlation function and the response:

$$\text{Im}\chi \sim \frac{\omega}{2T}\tilde{C}_{xx}(\omega)$$

In the DC limit, we get:

$$\text{Im}\chi = \frac{\omega}{2T}\tilde{C}_{xx}(\omega=0)$$

When:

$$\tilde{C}_{xx}(\omega=0) = \int_{-\infty}^{\infty}C_{xx}(t)dt = \int_{-\infty}^{\infty}\frac{1}{2}\left(\frac{t}{t_c}\right)^{-\frac{c}{\pi\eta^2}}dt$$

We define "phase transition" as When the response diverges. this will happen when $\frac{c}{\pi\eta^2}\leq 1$, so we get:

$$\eta_c = \sqrt{\frac{c}{\pi}}$$