Prof. E. Palti May 22, 2021

Particles 1 - Problem Set 3

(Due: Thursday, June 10th, 2021)

1 Spontaneous symmetry breaking in Non-Abelian Gauge Theories

(This problem is working through section 28.3.3 in Schwartz.)

Consider an $SO(3)$ gauge theory with scalars in the fundamental representation, a real 3-vector, ϕ_i:

$$\mathcal{L} = -\frac{1}{2}\mathrm{Tr}F^2 + \frac{1}{2}\left(D\phi\right)^2 + \frac{1}{2}m^2\left|\phi\right|^2 - \frac{\lambda}{4!}\left|\phi\right|^4 \tag{1}$$

The $SO(3)$ generators are given by

$$\tau^1 = i\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} \ , \ \tau^2 = i\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix} \ , \ \tau^3 = i\begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \ . \tag{2}$$

Note that the gauge transformation is purely real, allowing for ϕ to be real.

The ground state is at

$$\left|\phi\right| = v = \sqrt{\frac{6m^2}{\lambda}} \ , \tag{3}$$

and since the group $SO(3)$ is the group of rotations in 3 dimensions, we can always choose ϕ to take the form (**Bonus part**: Show this by an $SO(3)$ rotation.)

$$\langle\phi\rangle = \mathbf{v} = \begin{pmatrix} 0 \\ 0 \\ v \end{pmatrix} \ . \tag{4}$$

a) Show that the purely quadratic (non-derivative) term in the gauge fields is

$$\frac{g^2}{4}A^a_\mu A^b_\mu \mathbf{v}^T\left\{\tau^a, \tau^b\right\}\mathbf{v} \ . \tag{5}$$

b) How many gauge bosons gain a mass in the ground state? What is their mass?

2 Grand Unification

The standard Model gauge group is a product of three groups

$$G_{\rm SM} = SU(3) \otimes SU(2) \otimes U(1) \ . \tag{6}$$

Grand unified theories embed these into one group, the simplest of which is based on the group $SU(5)$.

a) The fundamental representation of $SU(5)$ is in terms of 24 generators of 5×5 complex matrices. The embedding must be such that the $SU(3)$, $SU(2)$ and $U(1)$ generators all commute with each other. We can embed the $U(1)$ hypercharge generator as

$$Y = \left(\begin{array}{ccc|cc} -\frac{1}{3} & 0 & 0 & 0 & 0 \\ 0 & -\frac{1}{3} & 0 & 0 & 0 \\ 0 & 0 & -\frac{1}{3} & 0 & 0 \\ \hline 0 & 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{2} \end{array} \right) \ , \tag{7}$$

where the λ^a are the generators of $SU(3)$.

what is the embedding of $SU(3)$ and $SU(2)$ into $SU(5)$ such that they commute with the hypercharge generator?

How many additional gauge fields beyond the Standard Model does $SU(5)$ unification predict?

b) The matters sector of the Standard Model (not including the right-handed neutrinos) is in the representations

$$(3,2)^L_{\frac{1}{6}} \ , \ \ (3,1)^R_{\frac{2}{3}} \ , \ \ (3,1)^R_{-\frac{1}{3}} \ , \ \ (1,2)^L_{-\frac{1}{2}} \ , \ \ (1,1)^R_{-1} \ , \tag{8}$$

where the superscripts L and R denote the chirality of the spinor. To embed the matter into $SU(5)$ it is convenient to write everything in terms of left-handed spinors only by acting with charge conjugation operator. To keep track of this we denote left-handed spinors that have been conjugated with a superscript c, so that

$$(\psi^c)^L = \left(\psi^R\right)^c \ . \tag{9}$$

In terms of the left-handed representations we have the Standard Model spectrum

$$(3,2)^L_{\frac{1}{6}} \ , \ \ (\bar{3},1)^L_{-\frac{2}{3}} \ , \ \ (\bar{3},1)^L_{\frac{1}{3}} \ , \ \ (1,2)^L_{-\frac{1}{2}} \ , \ \ (1,1)^L_{1} \ . \tag{10}$$

These fields are embedded into the fundamental representation of $SU(5)$ denoted χ_i, and the anti-symmetric representation ψ_{ij} - and anti-symmetric complex 5×5 matrix.

In a gauge transformation, The fundamental representation of the gauge generators T^a acts on the anti-symmetric representation of matter as

$$\chi \to T\chi + \chi T \ . \tag{11}$$

From the form of the action of the gauge generators, determine exactly how the (purely left-handed) fields are embedded into the above matter representations of $SU(5)$. Show in particular that the embedding reproduces the correct Hypercharge charges.

c) The Higgs doublet of the standard model must be embedded into the fundamental representation of $SU(5)$. But this predicts additional fields. What are the transformation properties of these fields under the Standard Model gauge symmetries?

d) The gauge group $SO(10)$ also contains the Standard Model Gauge group. It has a representation which is 16 dimensional (the spinor). Can this representation fit the whole Standard Model spectrum (of 1 generation)? Does this predict any additional fields? If so, are there natural candidates for them?

3 CP Violation in the Standard Model

The Lorentz group contains the Parity transformation

$$P \ : \ (t, \mathbf{x}) \to (t, -\mathbf{x}) \ . \tag{12}$$

In the Weyl basis, its action on spinors is

$$P \ : \ \psi \to \gamma_0 \psi \ . \tag{13}$$

Another important discrete action is charge conjugation, which acts as

$$C \ : \ \psi \to \psi^c = -i\gamma_2 \psi^* \ . \tag{14}$$

a) Show that the parity action (13) amounts to an exchange of left-handed with right-handed spinors

$$P \ : \ \psi_L \leftrightarrow \psi_R \ . \tag{15}$$

b) Show that the combination of charge conjugation and parity inversion CP, leads to the transformation properties

$$\bar{\psi}_i \psi_j \to \bar{\psi}_j \psi_i \ , \quad \bar{\psi}_i \gamma^5 \psi_j \to -\bar{\psi}_j \gamma^5 \psi_i \ , \tag{16}$$

$$\bar{\psi}_i \not{A} \psi_j \to \bar{\psi}_j \not{A} \psi_i \ , \quad \bar{\psi}_i \not{A} \gamma^5 \psi_j \to \bar{\psi}_j \not{A} \gamma^5 \psi_i \ , \tag{17}$$

with A_μ a gauge boson and $\not{A} \equiv A_\mu \gamma^\mu$.

c) Does the Standard Model violate Parity?

d) The coupling of the quarks to W-bosons in the mass basis is given in terms of the CKM matrix V as

$$\mathcal{L}_{\text{mass-basis}} = \frac{e}{\sqrt{2}\sin\theta_W}\left[W^+_\mu \bar{u}^i_L \gamma^\mu V^{ij} d^j_L + W^-_\mu \bar{d}^i_L \gamma^\mu \left(V^\dagger\right)^{ij} u^i_L\right] \ . \tag{18}$$

Show that the Standard Model is invariant under CP if and only if $V^* = V$.

Given the measurements of our universe, is the Standard Model invariant under CP?

e) **(bonus)** Another interesting discrete action is Time reversal

$$T \ : \ (t, \mathbf{x}) \to (-t, \mathbf{x}) \ . \tag{19}$$

Is the Standard Model invariant under the full combined action of CPT ?