# Tutorial 1 - Fourier Transform

## Introduction

1. Inner product:
   Given two integrable functions $f(x)$ and $g(x)$ on the interval $[-L, L]$, the *inner product* is defined as follows,

$$\boxed{\langle g, f\rangle \equiv \int_{-L}^{L} g^{*}(x) f(x)\, dx}.$$

2. Discrete Fourier series:
   Given an integrable function $f(x)$ on the interval $[-L, L]$, one may expand it into an infinite series as follows,

$$\boxed{f(x) = \frac{1}{\sqrt{2L}} \sum_{n=-\infty}^{\infty} c_n e^{i\frac{\pi n x}{L}}}.$$

   Defining the *harmonic basis* or *modes* as

$$\varphi_n \equiv \frac{1}{\sqrt{2L}} e^{i\frac{\pi n x}{L}},$$

   it is easy to see that this basis is *orthogonal* with respect to inner product, that is

$$\langle \varphi_m, \varphi_n\rangle = \frac{1}{2L}\int_{-L}^{L} e^{i\frac{\pi n x}{L}} e^{-i\frac{\pi m x}{L}}\, dx = \delta_{nm}.$$

   It can be easily seen by looking at the integrand as

$$e^{i\frac{\pi n x}{L}} e^{-i\frac{\pi m x}{L}} = \left(\cos\left[(n-m)\frac{\pi x}{L}\right] + i \sin\left[(n-m)\frac{\pi x}{L}\right]\right),$$

   which leads to

$$\begin{aligned} n \neq m: &\qquad \langle \varphi_m, \varphi_n\rangle \propto \cos\left[(n-m)\frac{\pi x}{L}\right] \text{ and } \sin\left[(n-m)\frac{\pi x}{L}\right]\Bigg|_{-L}^{L} = 0, \\ n = m: &\qquad \langle \varphi_m, \varphi_n\rangle = 1. \end{aligned}$$

   Using orthonormality of the basis functions we may calculate the *harmonic coefficients* as follows,

$$\begin{aligned} \frac{1}{\sqrt{2L}}\int_{-L}^{L} f(x) e^{-i\frac{\pi n x}{L}}\, dx &= \frac{1}{2L}\int_{-L}^{L} \sum_{m=-\infty}^{\infty} c_n e^{i\frac{\pi m x}{L}} e^{-i\frac{\pi n x}{L}}\, dx \\ &= \sum_{m=-\infty}^{\infty} c_m \frac{1}{2L}\int_{-L}^{L} e^{i\frac{\pi m x}{L}} e^{-i\frac{\pi n x}{L}}\, dx \\ &= \sum_{m=-\infty}^{\infty} c_m \delta_{nm} \end{aligned}$$

   thus,

$$\boxed{c_n = \frac{1}{\sqrt{2L}}\int_{-L}^{L} f(x) e^{-i\frac{\pi n x}{L}}\, dx}.$$

In addition, one can use the decomposition of the exponent and write

$$f(x) = a_0 + \sum_{n=1}^{\infty} \left[ a_n \cos \frac{\pi n x}{L} + b_n \sin \frac{\pi n x}{L} \right],$$

where

$$\begin{aligned} a_0 &= \frac{1}{\sqrt{2L}} \int_{-L}^{L} f(x)\, dx, \\ a_n &= \frac{1}{\sqrt{L}} \int_{-L}^{L} f(x) \cos \frac{\pi n x}{L} dx, \\ b_n &= \frac{1}{\sqrt{L}} \int_{-L}^{L} f(x) \sin \frac{\pi n x}{L} dx. \end{aligned}$$

3. Fourier transform:
   Given an integrable function $f(x)$ defined on all space, it is not possible to represent it as a discrete sum of harmonics, but rather as a continuous harmonic basis. In order to do that we take the continuous limit, by defining the mode *wavenumber* $k \equiv \pi n / L$ such that $\Delta k = \pi \Delta n / L$

$$f(x) = \frac{1}{\sqrt{2L}} \sum_n c_n e^{i\frac{\pi n x}{L}} \left( \frac{\Delta n}{1} \right) = \frac{1}{\sqrt{2L}} \sum_k \frac{L}{\pi} \left[ \frac{1}{\sqrt{2L}} \int_{-L}^{L} f(x') e^{-ikx'} dx' \right] e^{ikx} \Delta k = \frac{1}{\sqrt{2\pi}} \sum_k \underbrace{\left[ \frac{1}{\sqrt{2\pi}} \int_{-L}^{L} f(x') e^{-ikx'} dx' \right]}_{g(k)}$$

   and taking the limit of $L \to \infty$, $\Delta k \to dk$ and $\sum \to \int$, we find

$$\boxed{\begin{aligned} f(x) &= \tfrac{1}{\sqrt{2\pi}} \textstyle\int_{-\infty}^{\infty} g(k)\, e^{ikx} dk \\ g(k) &= \tfrac{1}{\sqrt{2\pi}} \textstyle\int_{-\infty}^{\infty} f(x)\, e^{-ikx} dx \end{aligned}},$$

   where $g(k) = \mathcal{F}[f(x)]$ is called the *Fourier transform* of $f(x)$.

4. Dirac's delta function:
   Writing the Fourier transform explicitly

$$\begin{aligned} f(x) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} g(k)\, e^{ikx} dk \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \left( \int_{-\infty}^{\infty} f(y)\, e^{-iky} dy \right) e^{ikx} dk \\ &= \int_{-\infty}^{\infty} f(y)\, dy \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ik(x-y)} dk, \end{aligned}$$

   we can define the Dirac delta function as

$$\boxed{\delta(x-y) \equiv \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ik(x-y)} dk},$$

   such that

$$\boxed{\int_{-\infty}^{\infty} f(y)\, \delta(x-y)\, dy = f(x)}.$$

   It is easy to show that

$$\delta(ax) = \frac{1}{|a|} \delta(x) \quad \text{and} \quad \delta(-x) = \delta(x).$$

5. Polynomial expansion:
In a similar manner to harmonic functions, one may expand a function $f(x)$, defined on $\Omega$, in terms of an orthonormal polynomial basis $P_n(x)$, which satisfies

$$\langle P_m(x), P_n(x)\rangle = \int_\Omega P_m^*(x) P_n(x) \omega(x) dx = \delta_{mn},$$

where $\omega(x)$ is a weight function that is introduced in order to normalize the basis (in harmonic functions $\omega(x) = 1$). Therefore

$$\boxed{f(x) = \sum_n a_n P_n(x), \quad x \in \Omega},$$

where

$$\boxed{a_n = \langle P_n(x), f(x)\rangle = \int_\Omega P_n^*(x) f(x) \omega(x) dx}.$$

The Dirac delta function is easily found from

$$\begin{aligned} f(x) &= \sum_n a_n P_n(x) \\ &= \int_\Omega f(y) dy \sum_n P_n(x) P_n^*(y) \omega(y) \end{aligned}$$

$$\boxed{\delta(x-y) = \sum_n P_n(x) P_n^*(y) \omega(y)}.$$

Some relevant polynomials:

| Name | Sym. | $\omega(x)$ | $\Omega$ |
|---|---|---|---|
| Lagendre | $\mathcal{L}_n(x)$ | 1 | $[-1, 1]$ |
| Hermite | $H_n(x)$ | $e^{-x^2}$ | $[-\infty, \infty]$ |
| Laguerre | $L_n^l(x)$ | $x^k e^{-x}$ | $[0, \infty]$ |

# Question 1:

Find the Fourier coefficients for the function $f(x) = x$ on the interval $[-\pi, \pi]$.
**Solution:**

Using the expansion to in Fourier modes

$$\varphi_n(x) = \frac{1}{\sqrt{2\pi}} e^{inx},$$

we find

$$c_n = \frac{1}{\sqrt{2\pi}} \int_{-\pi}^{\pi} x e^{-inx} dx.$$

Note that for $n = 0$ the integral is

$$c_0 = \frac{1}{\sqrt{2\pi}} \int_{-\pi}^{\pi} x dx = 0,$$

whereas for $n \neq 0$ we will use a useful trick for solving the integral by defining $\alpha = in$,

$$\begin{aligned} \int_{-\pi}^{\pi} x e^{-\alpha x} dx &= -\frac{d}{d\alpha} \int_{-\pi}^{\pi} e^{-\alpha x} dx \\ &= \frac{d}{d\alpha}\left[\frac{1}{\alpha}\left(e^{-\alpha\pi} - e^{\alpha\pi}\right)\right] \\ &= -\frac{1}{\alpha^2} \underbrace{\left(e^{-\alpha\pi} - e^{\alpha\pi}\right)}_{-2i\sin n\pi} + \frac{\pi}{\alpha} \underbrace{\left(-e^{-\alpha\pi} - e^{\alpha\pi}\right)}_{-2\cos n\pi}, \end{aligned}$$

then

$$c_n = \frac{1}{\sqrt{2\pi}} \frac{i}{n} \Bigg[ \underbrace{\cos n\pi}_{(-1)^n} - \underbrace{\frac{\sin n\pi}{n\pi}}_{0} \Bigg] = \frac{1}{\sqrt{2\pi}} \frac{i}{n} (-1)^n \quad \forall n \neq 0.$$

Therefore

$$\begin{aligned} f(x) &= \frac{1}{\sqrt{2\pi}} \sum_{n\neq 0} \frac{i}{n} (-1)^n e^{inx} \\ &= \frac{1}{\sqrt{2\pi}} \sum_{n=1}^{\infty} \frac{i}{n} (-1)^n \left( e^{inx} - e^{-inx} \right), \end{aligned}$$

$$\boxed{f(x) = \sum_{n=1}^{\infty} \frac{\sqrt{2}}{n\sqrt{\pi}} (-1)^{n+1} \sin nx}.$$

# Question 2:

Find the Fourier transform of the function

$$\varphi(x) = \begin{cases} c, & -a < x < a \\ 0 & \text{else} \end{cases}.$$

**Solution:**

Taking the Fourier transform of $\varphi(x)$ yields

$$\begin{aligned} \tilde{\varphi}(k) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \varphi(x) e^{-ikx} dx \\ &= \frac{1}{\sqrt{2\pi}} \int_{-a}^{a} c e^{-ikx} dx \\ &= \frac{c}{\sqrt{2\pi} ik} \left( e^{ika} - e^{-ika} \right) \end{aligned}$$

$$\boxed{\tilde{\varphi}(k) = c\sqrt{\frac{2}{\pi}} \frac{\sin ka}{k}}.$$

# Question 3:

Find the Fourier transform of the Gaussian function

$$g(x) = \frac{1}{\sigma} e^{-\frac{x^2}{2\sigma^2}}.$$

**Solution:**

Taking the Fourier transform of $g(x)$ yields

$$\begin{aligned} \tilde{g}(k) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} g(x) e^{-ikx} dx \\ &= \frac{1}{\sqrt{2\pi}\sigma} \int_{-\infty}^{\infty} e^{-\frac{x^2}{2\sigma^2}} e^{-ikx} dx. \end{aligned}$$

Looking at the power of the exponent we can write

$$-\frac{1}{2\sigma^2} \left( x^2 + 2i\sigma^2 kx \right) = -\frac{1}{2\sigma^2} \left( x^2 + 2i\sigma^2 kx - \sigma^4 k^2 \right) - \frac{\sigma^2 k^2}{2},$$

which leads to

$$\tilde{g}\left(k\right)=\frac{e^{-\sigma^{2}k^{2}/2}}{\sqrt{2\pi}\sigma}\underbrace{\int_{-\infty}^{\infty}e^{-\frac{u^{2}}{2\sigma^{2}}}du}_{I},$$

where we defined $u=x+i\sigma^{2}k$. we are left with a simple Gaussian integral $I$ which can be solved as follows

$$I^{2}=\left(\int_{-\infty}^{\infty}e^{-\frac{x^{2}}{2\sigma^{2}}}dx\right)^{2}=\int_{-\infty}^{\infty}e^{-\frac{x^{2}}{2\sigma^{2}}}dx\int_{-\infty}^{\infty}e^{-\frac{y^{2}}{2\sigma^{2}}}dy=\int_{-\infty}^{\infty}e^{-\frac{x^{2}+y^{2}}{2\sigma^{2}}}dxdy$$

moving to polar coordinates

$$I^{2}=\int_{0}^{2\pi}\int_{0}^{\infty}e^{-\frac{r^{2}}{2\sigma^{2}}}rdrd\varphi=\pi\int_{0}^{\infty}e^{-\frac{\xi}{2\sigma^{2}}}d\xi=2\pi\sigma^{2},$$

where we defined $\xi=r^{2}$ hence $d\xi=2rdr$, which gives $I=\sqrt{2\pi\sigma^{2}}$.
Therefore

$$\boxed{\tilde{g}\left(k\right)=e^{-\frac{\sigma^{2}k^{2}}{2}}}.$$

This is a Gaussian with width of $\tilde{\sigma}=1/\sigma$.