Tutorial 10 - Operators & Dirac Notation

Dirac Notation

Dirac proposed to separate the inner product notation, $\langle\alpha|\beta\rangle$, into *bra* $\langle\alpha|$ and *ket* $|\beta\rangle$. While the latter is a vector, the former is a linear function of vectors, i.e. when it eats a vector it poops a scalar. In a finite-dimensional vector space, if kets are represented by columns,

$$|\alpha\rangle \rightarrow \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{pmatrix},$$

then bras are rows

$$\langle\beta| \rightarrow \begin{pmatrix} \beta_1^* & \beta_2^* & \cdots & \beta_n^* \end{pmatrix},$$

so that

$$\langle\beta|\alpha\rangle = \beta_1^*\alpha_1 + \beta_2^*\alpha_2 + \cdots + \beta_n^*\alpha_n.$$

The collection of all bras constitutes another vector space called the *duel space*.

Question 1:

Consider a normalized vector $|\alpha\rangle$. We define the operator $\hat{P}_\alpha \equiv |\alpha\rangle\langle\alpha|$. When this operator acts on some vector it gives the component of the vector that lies along $|\alpha\rangle$:

$$\hat{P}_\alpha |\beta\rangle = \langle\alpha|\beta\rangle |\alpha\rangle .$$

We call such an operator a *projection operator* on the 1D space spanned by the vector $|\alpha\rangle$.

1. Show that $\hat{P}_\alpha^2 = \hat{P}_\alpha$. Find eigenvectors and eigenvalues of $\hat{P}_\alpha$.
2. Suppose the set $|e_j\rangle$, where $j = 1, 2, \ldots, n$, form an orthonormal basis to an n-dimensional vector space. Show that
$$\sum_{j=1}^{n} |e_j\rangle\langle e_j| = 1.$$
3. Suppose $\hat{Q}$ is an operator with a complete set of orthonormal eigenstates $\hat{Q}|e_j\rangle = \lambda_j |e_j\rangle$, where $j = 1, 2, \ldots, n$. Show that $\hat{Q}$ can be written in terms of its *spectral decomposition*,
$$\hat{Q} = \sum_{j=1}^{n} \lambda_j |e_j\rangle\langle e_j| .$$

Solution:

1. If $|\alpha\rangle$ is normalized then $\langle\alpha|\alpha\rangle = 1$. Thus
$$\boxed{\hat{P}_\alpha^2 = |\alpha\rangle \underbrace{\langle\alpha| |\alpha\rangle}_{1} \langle\alpha| = |\alpha\rangle\langle\alpha| = \hat{P}_\alpha}.$$

2. Let us call

$$\hat{A} = \sum_{j=1}^{n} |e_j\rangle \langle e_j| \,,$$

then acting with $\hat{A}$ on some state $|e_k\rangle$ yields

$$\hat{A}\,|e_k\rangle = \sum_{j=1}^{n} |e_j\rangle \langle e_j|e_k\rangle = \sum_{j=1}^{n} |e_j\rangle \, \delta_{jk} = |e_k\rangle \,,$$

which means that $\hat{A} = 1$, thus

$$\boxed{\sum_{j=1}^{n} |e_j\rangle \langle e_j| = 1}.$$

3. Let us act with $\hat{Q}$ on some composite state

$$\hat{Q}\sum_{i=1}^{n} c_i\,|e_i\rangle = \sum_{i=1}^{n} c_i\lambda_i\,|e_i\rangle = \sum_{i=1}^{n} \lambda_i\,|e_i\rangle \sum_{j=1}^{n} c_j\,\langle e_i|e_j\rangle = \sum_{i=1}^{n} \lambda_i\,|e_i\rangle\,\langle e_i| \sum_{j=1}^{n} c_j\,|e_j\rangle \quad \to \quad \boxed{\hat{Q} = \sum_{i=1}^{n} \lambda_i\,|e_i\rangle\,\langle e_i|}.$$

The Ladder Operators

Let us reconsider the harmonic oscillator. We may rewrite the time-independent Schrödinger equation as follows

$$\hat{H}\psi = \frac{1}{2m}\left[\hat{p}^2 + (m\omega\hat{x})^2\right]\psi = E\psi.$$

We would like to *factor* the Hamiltonian,

$$\hat{H} = \frac{1}{2m}\left[\hat{p}^2 + (m\omega\hat{x})^2\right],$$

in terms of $\hat{p}$ and $\hat{x}$, instead of the sum of operators. Had we been dealing with numbers, this would be easy

$$u^2 + v^2 = (v + iu)\,(v - iu)\,,$$

however, in our case $\hat{x}$ and $\hat{p}$ are operators that do not commute ($[\hat{x}, \hat{p}] = i\hbar$). But with this motivation we consider the composite operators

$$\hat{a}_\pm \equiv \frac{1}{\sqrt{2\hbar m\omega}}\,(m\omega\hat{x} \mp i\hat{p})\,,$$

such that

$$\hat{a}_-\hat{a}_+ = \frac{1}{2\hbar m\omega}\,(m\omega\hat{x} + i\hat{p})\,(m\omega\hat{x} - i\hat{p}) = \frac{1}{2\hbar m\omega}\left((m\omega\hat{x})^2 + im\omega\hat{p}\hat{x} - im\omega\hat{x}\hat{p} + \hat{p}^2\right) = \frac{1}{\hbar\omega}\hat{H} - \frac{i}{2\hbar}\,[\hat{x}, \hat{p}] = \frac{1}{\hbar\omega}\hat{H} + \frac{1}{2}.$$

Therefore, the Hamiltonian does not factor perfectly, leaving an extra $-1/2$:

$$\hat{H} = \hbar\omega\left(\hat{a}_-\hat{a}_+ - \frac{1}{2}\right).$$

Noting that

$$[\hat{a}_-, \hat{a}_+] = \frac{1}{2\hbar m\omega}\,[(m\omega\hat{x} + i\hat{p})\,, (m\omega\hat{x} - i\hat{p})] = \frac{i}{2\hbar}\,([\hat{p}, \hat{x}] - [\hat{x}, \hat{p}]) = \frac{i}{\hbar}\,[\hat{p}, \hat{x}] = 1,$$

we can also write

$$\hat{H} = \hbar\omega\left(\hat{a}_+\hat{a}_- + \frac{1}{2}\right).$$

In general, the Schrödinger equation reads

$$\hbar\omega\left(\hat{a}_\pm\hat{a}_\mp \pm \frac{1}{2}\right)\psi = E\psi.$$

Finally, let us check the commutation relation of $\hat{H}$ and $\hat{a}_\pm$:

$$\begin{aligned}\left[\hat{H},\hat{a}_\pm\right] &= \hbar\omega\left[\left(\hat{a}_+\hat{a}_- + \frac{1}{2}\right),\hat{a}_\pm\right]\\ &= \hbar\omega\left[\hat{a}_+\hat{a}_-,\hat{a}_\pm\right]\\ &= \hbar\omega\left(\hat{a}_+\left[\hat{a}_-,\hat{a}_\pm\right] + \left[\hat{a}_+,\hat{a}_\pm\right]\hat{a}_-\right),\end{aligned}$$

thus

$$\left[\hat{H},\hat{a}_+\right] = \hbar\omega\hat{a}_+ \quad \text{and} \quad \left[\hat{H},\hat{a}_-\right] = -\hbar\omega\hat{a}_-.$$

Therefore, it is clear that $\hat{a}_\pm\psi$ are also eigenstates of $\hat{H}$:

$$\begin{aligned}\hat{H}\left(\hat{a}_+\psi\right) &= \left(\hat{a}_+\hat{H} + \hbar\omega\hat{a}_+\right)\psi = \left(E+\hbar\omega\right)\hat{a}_+\psi,\\ \hat{H}\left(\hat{a}_-\psi\right) &= \left(\hat{a}_-\hat{H} - \hbar\omega\hat{a}_-\right)\psi = \left(E-\hbar\omega\right)\hat{a}_-\psi.\end{aligned}$$

We call $\hat{a}_\pm$ the *ladder operators*, because they allow us to go up and down in energy: $\hat{a}_+$ is the *raising operator* and $\hat{a}_-$ is the *lowering operator*. Noting that

$$\left[\hat{a}_-,\hat{a}_+\right]^\dagger = \left[\hat{a}_+^\dagger,\hat{a}_-^\dagger\right] = \left[\hat{a}_-,\hat{a}_+\right] \quad \rightarrow \quad \hat{a}_+ = \hat{a}_-^\dagger,$$

it is common to define the lowering operator as $\hat{a}_- \equiv \hat{a}$, so that $\hat{a}^\dagger$ is the raising operator.

However, there must be a ground state, physically. Thus, we must require some state ψ_0 which would follow

$$\hat{a}\psi_0 = 0.$$

This condition yields

$$\frac{1}{\sqrt{2\hbar m\omega}}\left(m\omega\hat{x} + i\hat{p}\right)\psi_0 = 0 \quad \rightarrow \quad \frac{d\psi_0}{dx} = -\frac{m\omega}{\hbar}x\psi_0 \quad \rightarrow \quad \psi_0\left(x\right) = Ae^{-\frac{m\omega}{2\hbar}x^2}.$$

Requiring ψ_0 to be normalized,

$$|A|^2\int_{-\infty}^{\infty} e^{-\frac{m\omega}{\hbar}x^2}dx = |A|^2\sqrt{\frac{\pi\hbar}{m\omega}} = 1 \quad \rightarrow \quad \boxed{\psi_0\left(x\right) = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} e^{-\frac{m\omega}{2\hbar}x^2}}.$$

Now, to determine the energy of the ground state we use

$$\hat{H}\psi_0 = \hbar\omega\left(\hat{a}^\dagger\hat{a} + \frac{1}{2}\right)\psi_0 = \frac{1}{2}\hbar\omega\psi_0 \quad \rightarrow \quad \boxed{E_0 = \frac{1}{2}\hbar\omega}.$$

In order to generate all the other eigenstates we only need to act on ψ_0 with $\hat{a}_+$ successively, in which case

$$\boxed{\psi_n\left(x\right) = A_n\left(\hat{a}^\dagger\right)^n\psi_0, \quad \text{with} \quad E_n = \hbar\omega\left(n + \frac{1}{2}\right)},$$

where A_n is a normalization factor.
In the Dirac notation we write $\psi_n \rightarrow |n\rangle$, so that

$$\hat{H}\left|n\right\rangle = E_n\left|n\right\rangle = \hbar\omega\left(\hat{a}^\dagger\hat{a} + \frac{1}{2}\right)\left|n\right\rangle, \quad \text{with} \quad \left|n\right\rangle = \left(\hat{a}^\dagger\right)^n\left|0\right\rangle.$$

When acting with a ladder operator on a state $|n\rangle$ we get

$$\hat{a}^\dagger\left|n\right\rangle = c_n\left|n+1\right\rangle, \quad \text{and} \quad \hat{a}\left|n\right\rangle = d_n\left|n-1\right\rangle.$$

Noting that

$$\langle m|\hat{a}|n\rangle = \langle\hat{a}^\dagger m|n\rangle \quad \rightarrow \quad d_n\left\langle m|n-1\right\rangle = c_m^*\left\langle m+1|n\right\rangle,$$

we can write

$$|d_n|^2 = \langle n|\hat{a}^\dagger\hat{a}|n\rangle = n,$$
$$|c_n|^2 = \langle n|\hat{a}\hat{a}^\dagger|n\rangle = n+1,$$

where we used the relations

$$\hat{a}^\dagger\hat{a}\,|n\rangle = \left(\frac{1}{\hbar\omega}\hat{H} - \frac{1}{2}\right)|n\rangle = \left(n + \frac{1}{2} - \frac{1}{2}\right)|n\rangle = n\,|n\rangle\,,$$
$$\hat{a}\hat{a}^\dagger\,|n\rangle = \left(\frac{1}{\hbar\omega}\hat{H} + \frac{1}{2}\right)|n\rangle = (n+1)\,|n\rangle\,.$$

Therefore

$$\boxed{\hat{a}^\dagger\,|n\rangle = \sqrt{n+1}\,|n+1\rangle\,, \quad \text{and} \quad \hat{a}\,|n\rangle = \sqrt{n}\,|n-1\rangle}\,.$$

Thus

$$|1\rangle = \hat{a}^\dagger\,|0\rangle\,, \qquad |3\rangle = \tfrac{1}{\sqrt{2\times3}}\left(\hat{a}^\dagger\right)^3|0\rangle\,,$$
$$|2\rangle = \tfrac{1}{\sqrt{2}}\left(\hat{a}^\dagger\right)^2|0\rangle\,, \qquad |4\rangle = \tfrac{1}{\sqrt{2\times3\times4}}\left(\hat{a}^\dagger\right)^4|0\rangle\,,$$

and in general $\boxed{A_n = 1/\sqrt{n!}}$.

In general, assuming that two operators have the following commutation relation

$$\left[\hat{Q},\hat{X}\right] = -c\hat{X},$$

where c is some complex constant, and $|n\rangle$ are the non-degenerate eigenstates of $\hat{Q}$,

$$\hat{Q}\,|n\rangle = n\,|n\rangle\,,$$

then

$$\hat{Q}\hat{X}\,|n\rangle = (n-c)\,\hat{X}\,|n\rangle\,,$$

hence $\hat{X}\,|n\rangle$ is an eigenstate of $\hat{Q}$ with eigenvalue $n-c$. However,

$$\hat{Q}\,|n-c\rangle = (n-c)\,|n-c\rangle\,,$$

and the eigenstates are non-degenerate, thus

$$\hat{X}\,|n\rangle = |n-c\rangle\,,$$

and we call $\hat{X}$ the lowering (raising) operator if c is positive (negative). In the case of $\hat{Q}$ being Hermitian, c is real and we have$\hat{X}^\dagger$ as the raising (lowering) operator.

Question 2:

Consider a simple harmonic oscillator with mass m and frequency ω. At time $t = 0$ the particle is at a state

$$|\psi\rangle = \frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle .$$

1. Find the state of the particle at time $t > 0$.
2. Calculate $\langle x\rangle(t)$, $\langle p\rangle(t)$ and $\langle H\rangle(t)$.
3. Show that the uncertainty principle holds.

Solution:

1. The energy of state $|n\rangle$ is $E_n = \hbar\omega(n + 1/2)$, thus

$$\boxed{|\Psi\rangle = \frac{1}{\sqrt{2}}\left(e^{-i\omega t/2}|0\rangle + e^{-i3\omega t/2}|1\rangle\right)}.$$

2. The expectation value of an operator $\hat{Q}$ is

$$\langle Q\rangle = \frac{1}{2}\left(\left\langle 0|\hat{Q}|0\right\rangle + e^{-i\omega t}\left\langle 0|\hat{Q}|1\right\rangle + e^{i\omega t}\left\langle 1|\hat{Q}|0\right\rangle + \left\langle 1|\hat{Q}|1\right\rangle\right)$$

We can write the operators $\hat{x}$ and $\hat{p}$ in terms of $\hat{a}$,

$$\hat{a} = \frac{1}{\sqrt{2m\hbar\omega}}\left(m\omega\hat{x} + i\hat{p}\right) \quad \rightarrow \quad \begin{aligned} \hat{x} &= \sqrt{\tfrac{\hbar}{2m\omega}}\left(\hat{a}^\dagger + \hat{a}\right), \\ \hat{p} &= i\sqrt{\tfrac{m\hbar\omega}{2}}\left(\hat{a}^\dagger - \hat{a}\right). \end{aligned}$$

Thus

$$\begin{aligned} \langle k|\hat{x}|n\rangle &= \sqrt{\frac{\hbar}{2m\omega}}\langle k|\left(\hat{a}^\dagger + \hat{a}\right)|n\rangle = \sqrt{\frac{\hbar}{2m\omega}}\left(\sqrt{n+1}\langle k|n+1\rangle + \sqrt{n}\langle k|n-1\rangle\right) \\ &= \sqrt{\frac{\hbar}{2m\omega}}\left(\sqrt{n+1}\delta_{k,n+1} + \sqrt{n}\delta_{k,n-1}\right), \\ \langle k|\hat{p}|n\rangle &= i\sqrt{\frac{\hbar m\omega}{2}}\langle k|\left(\hat{a}^\dagger - \hat{a}\right)|n\rangle = i\sqrt{\frac{\hbar m\omega}{2}}\left(\sqrt{n+1}\delta_{k,n+1} - \sqrt{n}\delta_{k,n-1}\right). \end{aligned}$$

Therefore

$$\boxed{\langle x\rangle(t) = \frac{1}{2}\left(e^{i\omega t}\langle 1|\hat{x}|0\rangle + e^{-i\omega t}\langle 0|\hat{x}|1\rangle\right) = \frac{1}{2}\sqrt{\frac{\hbar}{2m\omega}}\left(e^{i\omega t} + e^{-i\omega t}\right) = \sqrt{\frac{\hbar}{2m\omega}}\cos\omega t},$$

$$\boxed{\langle p\rangle(t) = \frac{1}{2}\left(e^{i\omega t}\langle 1|\hat{p}|0\rangle + e^{-i\omega t}\langle 0|\hat{p}|1\rangle\right) = \frac{i}{2}\sqrt{\frac{m\hbar\omega}{2}}\left(e^{i\omega t} - e^{-i\omega t}\right) = -\sqrt{\frac{m\hbar\omega}{2}}\sin\omega t = m\frac{d\langle x\rangle}{dt}}.$$

Finally,

$$\left\langle k|\hat{H}|n\right\rangle = \hbar\omega\left\langle k|\left(\hat{a}^\dagger\hat{a} + \frac{1}{2}\right)|n\right\rangle = \hbar\omega\left(n + \frac{1}{2}\right)\delta_{kn},$$

thus

$$\boxed{\langle H\rangle = \frac{1}{2}\left(\left\langle 0|\hat{H}|0\right\rangle + \left\langle 1|\hat{H}|1\right\rangle\right) = \hbar\omega}.$$

3. We need to use

$$\hat{x}^2 = \frac{\hbar}{2m\omega}\left(\hat{a}^\dagger\hat{a}^\dagger + \hat{a}^\dagger\hat{a} + \hat{a}\hat{a}^\dagger + \hat{a}\hat{a}\right) = \frac{\hbar}{2m\omega}\left(\hat{a}^\dagger\hat{a}^\dagger + \hat{a}\hat{a} + 2\hat{a}^\dagger\hat{a} + 1\right),$$

and

$$\hat{p}^2 = 2m\hat{H} - (m\omega)^2\,\hat{x}^2 = \frac{m\hbar\omega}{2}\left(2\hat{a}^\dagger\hat{a} + 1 - \hat{a}^\dagger\hat{a}^\dagger - \hat{a}\hat{a}\right).$$

Thus, in generalr

$$\begin{aligned}\langle k|\hat{x}^2|n\rangle &= \frac{\hbar}{2m\omega}\left(\sqrt{(n+1)(n+2)}\delta_{k,n+2} + \sqrt{n(n-1)}\delta_{k,n-2} + (2n+1)\,\delta_{k,n}\right),\\ \langle k|\hat{p}^2|n\rangle &= \frac{m\hbar\omega}{2}\left((2n+1)\,\delta_{k,n} - \sqrt{(n+1)(n+2)}\delta_{k,n+2} - \sqrt{n(n-1)}\delta_{k,n-2}\right).\end{aligned}$$

In our case

$$\langle x^2\rangle = \frac{\hbar}{m\omega} \quad \text{and} \quad \langle p^2\rangle = m\hbar\omega,$$

thus

$$\sigma_x = \sqrt{\frac{\hbar}{m\omega} - \frac{\hbar}{2m\omega}\cos^2\omega t} \geq \sqrt{\frac{\hbar}{2m\omega}} \quad \text{and} \quad \sigma_p = \sqrt{m\hbar\omega - \frac{m\hbar\omega}{2}\sin^2\omega t} \geq \sqrt{\frac{m\hbar\omega}{2}}$$

$$\boxed{\sigma_x\sigma_p \geq \frac{\hbar}{2}}.$$

Matrix Elements

Just as states can be represented by vectors with respect to some basis

$$\begin{aligned} |\alpha\rangle &= \textstyle\sum_n a_n |e_n\rangle\,, \qquad & |\beta\rangle &= \textstyle\sum_n b_n |e_n\rangle\,, \\ a_n &= \langle e_n|\alpha\rangle\,, & b_n &= \langle e_n|\beta\rangle\,, \end{aligned}$$

operators can be represented by their *matrix elements*

$$Q_{mn} \equiv \left\langle e_m|\hat{Q}|e_n\right\rangle,$$

so that

$$\text{if} \quad |\beta\rangle = \hat{Q}\,|\alpha\rangle\,, \quad \text{then} \quad \sum_n b_n\,|e_n\rangle = \sum_n a_n\hat{Q}\,|e_n\rangle \quad \to \quad b_m = \sum_n Q_{mn}a_n.$$

Thus, Q_{mn} contain the information about how the components transform.

Question 3:

Consider a simple harmonic oscillator with mass m and frequency ω. Calculate the representative matrix elements for the operators $\hat{x}$ and $\hat{p}$, in the basis of the energy states.

Solution:

The matrix elements of an operator $\hat{Q}$ in the basis of the energy states $\{|n\rangle\}$ are

$$Q_{kn} = \left\langle k|\hat{Q}|n\right\rangle.$$

We can express $\hat{x}$ and $\hat{p}$ in terms of a and $a^\dagger$ so that

$$x_{kn} = \sqrt{\frac{\hbar}{2m\omega}}\,\langle k|\left(\hat{a}^\dagger + \hat{a}\right)|n\rangle = \sqrt{\frac{\hbar}{2m\omega}}\left(\sqrt{n+1}\,\langle k|n+1\rangle + \sqrt{n}\,\langle k|n-1\rangle\right) = \sqrt{\frac{\hbar}{2m\omega}}\left(\sqrt{n+1}\delta_{k,n+1} + \sqrt{n}\delta_{k,n-1}\right),$$

so that

$$\boxed{\hat{x} \to \sqrt{\frac{\hbar}{2m\omega}}\begin{pmatrix} 0 & 1 & & & 0 \\ 1 & 0 & \sqrt{2} & & \\ & \sqrt{2} & 0 & \sqrt{3} & \\ & & \sqrt{3} & 0 & \sqrt{4} \\ 0 & & & \sqrt{4} & \ddots \end{pmatrix}}.$$

While

$$p_{kn} = i\sqrt{\frac{m\hbar\omega}{2}}\,\langle k|\left(\hat{a}^\dagger - \hat{a}\right)|n\rangle = i\sqrt{\frac{m\hbar\omega}{2}}\left(\sqrt{n+1}\,\langle k|n+1\rangle - \sqrt{n}\,\langle k|n-1\rangle\right) = i\sqrt{\frac{m\hbar\omega}{2}}\left(\sqrt{n+1}\delta_{k,n+1} - \sqrt{n}\delta_{k,n-1}\right),$$

so that

$$\boxed{\hat{p} \to i\sqrt{\frac{m\hbar\omega}{2}}\begin{pmatrix} 0 & -1 & & & 0 \\ 1 & 0 & -\sqrt{2} & & \\ & \sqrt{2} & 0 & -\sqrt{3} & \\ & & \sqrt{3} & 0 & -\sqrt{4} \\ 0 & & & \sqrt{4} & \ddots \end{pmatrix}}.$$

Question 4:

Consider a simple harmonic oscillator with mass m and frequency ω. Calculate $\langle x\rangle$, $\langle x^2\rangle$, $\langle p\rangle$ and $\langle p^2\rangle$. What state will give the lowest uncertainty $\sigma_x\sigma_p$?

Solution:

We've already seen that

$$\begin{aligned}
\langle m|\hat{x}|n\rangle &= \sqrt{\frac{\hbar}{2m\omega}}\left(\sqrt{n+1}\delta_{m,n+1} + \sqrt{n}\delta_{m,n-1}\right), \\
\langle m|\hat{p}|n\rangle &= i\sqrt{\frac{\hbar m\omega}{2}}\left(\sqrt{n+1}\delta_{m,n+1} - \sqrt{n}\delta_{m,n-1}\right), \\
\left\langle m|\hat{x}^2|n\right\rangle &= \frac{\hbar}{2m\omega}\left(\sqrt{(n+1)(n+2)}\delta_{m,n+2} + \sqrt{n(n-1)}\delta_{m,n-2} + (2n+1)\delta_{m,n}\right), \\
\left\langle m|\hat{p}^2|n\right\rangle &= \frac{m\hbar\omega}{2}\left((2n+1)\delta_{m,n} - \sqrt{(n+1)(n+2)}\delta_{m,n+2} - \sqrt{n(n-1)}\delta_{m,n-2}\right).
\end{aligned}$$

Therefore, it is straightforward to plug $n = m$ and get

$$\boxed{\langle x\rangle = \langle p\rangle = 0},$$

and

$$\boxed{\left\langle x^2\right\rangle = \frac{\hbar}{2m\omega}(2n+1) \quad \text{and} \quad \left\langle p^2\right\rangle = \frac{m\hbar\omega}{2}(2n+1)},$$

thus,

$$\sigma_x\sigma_p = \sqrt{\langle x^2\rangle\langle p^2\rangle} = \frac{\hbar}{2}(2n+1),$$

with the lowest uncertainty at $\boxed{n=0}$ (ψ_0 is a Gaussian).

Question 5:

Consider a system with two linearly independent states

$$|1\rangle \to \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{and} \quad |2\rangle \to \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

A general state is a normalized linear combination of the two

$$|\mathcal{S}\rangle = a|1\rangle + b|2\rangle \to \begin{pmatrix} a \\ b \end{pmatrix}, \quad \text{with} \quad |a|^2 + |b|^2 = 1.$$

Since the Hamiltonian is a Hermitian operator (observable) it can be represented by the matrix

$$H = \begin{pmatrix} h & g \\ g & h \end{pmatrix}, \quad \text{with} \quad \text{Im}[h] = \text{Im}[g] = 0.$$

If the system starts out in state $|1\rangle$, what is its state at time t?

Solution:

The time-dependent Schrödinger equation reads

$$i\hbar\frac{d}{dt}|\mathcal{S}(t)\rangle = \hat{H}|\mathcal{S}(t)\rangle.$$

Thus, we first need to find the eigenvectors and eigenfunctions of $\hat{H}$, which is done via the time-independent Schrödinger equation

$$\hat{H}|s\rangle = E|s\rangle,$$

for which the characteristic equation reads

$$\det\begin{pmatrix} h-E & g \\ g & h-E \end{pmatrix} = (h-E)^2 - g^2 = 0 \quad \rightarrow \quad h-E = \mp g \quad \rightarrow \quad \boxed{E_\pm = h \pm g}.$$

The eigenvectors that correspond to $E_\pm$ are determined by the equation of the eigenvalues

$$\begin{pmatrix} h & g \\ g & h \end{pmatrix}\begin{pmatrix} \alpha \\ \beta \end{pmatrix} = (h \pm g)\begin{pmatrix} \alpha \\ \beta \end{pmatrix} \quad \rightarrow \quad h\alpha + g\beta = (h \pm g)\,\alpha \quad \rightarrow \quad \beta = \pm\alpha,$$

so that the normalized eigenvectors are

$$\boxed{|s_\pm\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ \pm 1 \end{pmatrix}}.$$

Next we expand the initial state as a combination of $|s_\pm\rangle$

$$|\mathcal{S}(0)\rangle = |1\rangle = (a\,|s_+\rangle + b\,|s_-\rangle) \rightarrow \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} a+b \\ a-b \end{pmatrix} \quad \rightarrow \quad a = b = \frac{1}{\sqrt{2}}.$$

Finally, we add the time dependence to get the time dependent state

$$\boxed{|\mathcal{S}(t)\rangle = \frac{1}{\sqrt{2}}\left[e^{-i(h+g)t/\hbar}\,|s_+\rangle + e^{-i(h-g)t/\hbar}\,|s_-\rangle\right] \rightarrow \frac{1}{2}e^{-iht/\hbar}\left[e^{-igt/\hbar}\begin{pmatrix} 1 \\ 1 \end{pmatrix} + e^{igt/\hbar}\begin{pmatrix} 1 \\ -1 \end{pmatrix}\right] = e^{-iht/\hbar}\begin{pmatrix} \cos(gt/\hbar) \\ -i\sin(gt/\hbar) \end{pmatrix}}.$$