Tutorial 11 - Central Force & Angular Momentum

Central Force

The Schrödinger equation reads

$$i\hbar\frac{\partial\Psi}{\partial t} = \hat{H}\Psi,$$

which in three dimensions can be written as

$$i\hbar\frac{\partial\Psi}{\partial t} = -\frac{\hbar^2}{2m}\nabla^2\Psi + V\left(\boldsymbol{r}\right)\Psi.$$

Now V and Ψ are functions of the position vector $\boldsymbol{r}$ and t, so that the probability to find the particle in the infinitesimal volume $d^3\boldsymbol{r}$ is $\left|\Psi\left(\boldsymbol{r},t\right)\right|^2 d^3\boldsymbol{r}$, which is yields 1 when integrated over all space.
The most common potentials we encounter in physics are the *central potentials*, for which $V\left(\boldsymbol{r}\right) = V\left(r\right)$.
In spherical coordinates, the time-independent Schrödinger equation reads

$$-\frac{\hbar^2}{2m}\left[\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\psi}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\psi}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2\psi}{\partial\phi^2}\right] + V\psi = E\psi.$$

Angular Momentum

In classical mechanics of motion in central forces the angular momentum is a conserved quantity, thus it is no surprise that it plays an important role in quantum mechanics as well. Starting from the classical definition $\boldsymbol{L} = \boldsymbol{r}\times\boldsymbol{p}$, and promoting the classical quantities into operators we have

$$\hat{L}_x = \hat{y}\hat{p}_z - \hat{z}\hat{p}_y, \quad \hat{L}_y = \hat{z}\hat{p}_x - \hat{x}\hat{p}_z, \quad \hat{L}_z = \hat{x}\hat{p}_y - \hat{y}\hat{p}_x,$$

where $\hat{p}_q = -i\hbar\partial/\partial q$. In what follows we will drop the hat notation for operators for simplicity.

Question 1:

Calculate the following commutation relations:

- $\left[\hat{L}_i, \hat{r}_j\right]$
- $\left[\hat{L}_i, \hat{p}_j\right]$
- $\left[\hat{L}_i, \hat{L}_j\right]$
- $\left[\hat{L}^2, \hat{L}_i\right]$

Solution:

Let us use the index notation, so that

$$L_i = \varepsilon_{ijk}r_jp_k.$$

- $\left[L_i, r_j\right] = \left[\varepsilon_{ikl}r_kp_l, r_j\right] = \varepsilon_{ikl}r_k\left[p_l, r_j\right] = -\varepsilon_{ikl}r_ki\hbar\delta_{lj} = i\hbar\varepsilon_{ijk}r_k.$
- $\left[L_i, p_j\right] = \left[\varepsilon_{ikl}r_kp_l, p_j\right] = \varepsilon_{ikl}p_l\left[r_k, p_j\right] = \varepsilon_{ikl}r_ki\hbar\delta_{kj} = i\hbar\varepsilon_{ijl}p_l.$

- $$\begin{aligned}
[L_i, L_j] &= [\varepsilon_{ikl} r_k p_l, \varepsilon_{jnm} r_n p_m] \\
&= \varepsilon_{ikl}\varepsilon_{jnm} \left(r_k [p_l, r_n] p_m + r_n [r_k, p_m] p_l\right) \\
&= i\hbar\varepsilon_{ikl}\varepsilon_{jnm} \left(-r_k p_m \delta_{ln} + r_n p_l \delta_{km}\right) \\
&= i\hbar \left(-\varepsilon_{ikl}\varepsilon_{jlm} r_k p_m + \varepsilon_{ikl}\varepsilon_{jnk} r_n p_l\right) \\
&= i\hbar \left[\left(\delta_{ij}\delta_{km} - \delta_{im}\delta_{kj}\right) r_k p_m + \left(\delta_{lj}\delta_{in} - \delta_{ij}\delta_{ln}\right) r_n p_l\right] \\
&= i\hbar \left(\cancel{\delta_{ij} r_k p_k} - r_j p_i + r_i p_j - \cancel{\delta_{ij} r_l p_l}\right) \\
&= i\hbar\varepsilon_{ijk} r_i p_j,
\end{aligned}$$

  thus

  $$\boxed{[L_i, L_j] = i\hbar\varepsilon_{ijk} L_k}.$$

- Writing $L^2 = L_j L_j$, we have

  $$\begin{aligned}
\left[L^2, L_i\right] &= [L_j L_j, L_i] \\
&= L_j [L_j, L_i] + [L_j, L_i] L_j \\
&= i\hbar \left(\varepsilon_{jik} L_j L_k + \varepsilon_{jik} L_k L_j\right) \\
&= i\hbar \left(\varepsilon_{jik} + \varepsilon_{kij}\right) L_j L_k \\
&= i\hbar \left(\varepsilon_{jik} - \varepsilon_{jik}\right) L_j L_k,
\end{aligned}$$

  thus

  $$\boxed{\left[L^2, L_i\right] = 0}.$$

The Ladder Operators

Therefore, we may be able to find a shared eigenbasis $\{|\psi\rangle\}$ for L and L_z (it is customary to use the z component):

$$L^2 |\psi\rangle = \lambda |\psi\rangle \quad \text{and} \quad L_z |\psi\rangle = \mu |\psi\rangle .$$

In the same way as we did in the case of the harmonic oscillator with p^2 and x^2, we will define

$$L_\pm \equiv L_x \pm i L_y,$$

for which

$$[L_z, L_\pm] = [L_z, L_x] \pm i [L_z, L_y] = \hbar (i L_y \pm L_x) = \pm\hbar L_\pm \quad \text{and} \quad \left[L^2, L_\pm\right] = 0.$$

Using these two relations, we may write

$$L^2 L_\pm |\psi\rangle = L_\pm L^2 |\psi\rangle = \lambda L_\pm |\psi\rangle \quad \text{and} \quad L_z L_\pm |\psi\rangle = (L_\pm L_z \pm \hbar L_\pm) |\psi\rangle = (\mu \pm \hbar) L_\pm |\psi\rangle ,$$

thus $L_\pm |\psi\rangle$ is an eigenstate of both L^2 and L_z with eigenvalues λ and $(\mu \pm \hbar)$, correspondingly. Thus, $L_\pm$ is called the *raising (lowering) operator,* as it increases and lowers the eigenvalue of L_z by $\hbar$. However, since $\lambda \leq \mu^2$ we must require that there is a top limit to this raising process:

$$L_+ |\psi_t\rangle = 0.$$

Denoting the eigenvalue of L_z at $|\psi_t\rangle$ by $\hbar\ell$, we have

$$L_z |\psi_t\rangle = \hbar\ell |\psi_t\rangle \quad \text{and} \quad L^2 |\psi_t\rangle = \lambda |\psi_t\rangle .$$

In order to relate $L_\pm$ to L^2 we calculate

$$\begin{aligned}
L_\pm L_\mp &= (L_x \pm i L_y)(L_x \mp i L_y) \\
&= L_x^2 \pm i L_y L_x \mp i L_x L_y + L_y^2 \\
&= L^2 - L_z^2 \mp i [L_x, L_y] \\
&= L^2 - L_z^2 \pm \hbar L_z,
\end{aligned}$$

or

$$L^2 = L_\pm L_\mp + L_z^2 \mp \hbar L_z.$$

Thus

$$L^2 |\psi_t\rangle = \left(L_- L_+ + L_z^2 + \hbar L_z\right) |\psi_t\rangle = \hbar^2 \ell (\ell+1) |\psi_t\rangle = \lambda |\psi_t\rangle \quad \rightarrow \quad \lambda = \hbar^2 \ell (\ell+1).$$

Carrying a similar calculation for the bottom limit $L_- |\psi_b\rangle = 0$, for which $L_z |\psi_b\rangle = \hbar\bar{\ell} |\psi_b\rangle$, we find $\lambda = \hbar^2 \bar{\ell}\left(\bar{\ell} - 1\right)$. Comparing the expressions for λ in terms of the max and min eigenvalues of L_z we get either $\bar{\ell} = \ell + 1$, which is nonsense (the lowest eigenvalue of L_z cannot be larger than the largest eigenvalue), or

$$\bar{\ell} = -\ell.$$

So that the eigenvalues of L_z, denoted by $m\hbar$ (not to be confused with the mass of the particle), ranging between $-\ell$ and ℓ in N integer steps, thus ℓ must be an integer or half-integer. It is thus common to denote the eigenstates of L^2 and L_z by $|\ell, m\rangle$ so that

$$L^2 |\ell, m\rangle = \hbar^2 \ell (\ell+1) |\ell, m\rangle \quad \text{and} \quad L_z |\ell, m\rangle = \hbar m |\ell, m\rangle,$$

where

$$\ell = 0, 1/2, 1, 3/2, \ldots \quad \text{and} \quad m = -\ell, -\ell+1, \ldots \ell-1, \ell.$$

We will often act with $L_\pm$ on an eigenstate $|\ell, m\rangle$, in which case we expect

$$L_+ |\ell, m\rangle = A_{\ell,m} |\ell, m+1\rangle \quad \text{and} \quad L_- |\ell, m\rangle = B_{\ell,m} |\ell, m-1\rangle.$$

In order to find these constants, let us first note that, since L_x L_y and L_z are hermitian (for example $L_z^\dagger = (xp_y - yp_x)^\dagger = p_y^\dagger x^\dagger - p_x^\dagger y^\dagger = xp_y - yp_x = L_z$), then

$$L_\pm^\dagger = (L_x \pm iL_y)^\dagger = L_\mp.$$

Now, taking the inner product

$$\langle \ell, m | L_- L_+ | \ell, m\rangle = |A_{\ell,m}|^2 \langle \ell, m+1 | \ell, m+1\rangle = |A_{\ell,m}|^2,$$

on the other hand

$$\langle \ell, m | L_- L_+ | \ell, m\rangle = \langle \ell, m | \left(L^2 - L_z^2 - \hbar L_z\right) | \ell, m\rangle = \hbar^2 \ell (\ell+1) - \hbar^2 m (m+1).$$

Therefore

$$A_{\ell,m} = \hbar\sqrt{\ell(\ell+1) - m(m+1)} = \hbar\sqrt{(\ell-m)(\ell+m+1)}.$$

In the same manner, we may calculate $\langle \ell, m | L_+ L_- | \ell, m\rangle$ to find

$$B_{\ell,m} = \hbar\sqrt{(\ell+m)(\ell-m+1)}.$$

Therefore

$$L_+ |\ell, m\rangle = \hbar\sqrt{(\ell-m)(\ell+m+1)} |\ell, m+1\rangle,$$
$$L_- |\ell, m\rangle = \hbar\sqrt{(\ell+m)(\ell-m+1)} |\ell, m-1\rangle.$$

Eigenfunctions

We can write L^2, L_z and $L_\pm$ in the spherical position basis ($\langle \theta, \phi | \ell, m\rangle$) as

$$L_z = -i\hbar\frac{\partial}{\partial\phi},$$
$$L_\pm = \pm\hbar e^{\pm i\phi}\left(\frac{\partial}{\partial\theta} \pm i\cot\theta\frac{\partial}{\partial\phi}\right),$$

and, using $L^2 = L_\pm L_\mp + L_z^2 \mp \hbar L_z$,

$$L^2 = -\hbar^2 \left[\frac{1}{\sin\theta} \frac{\partial}{\partial\theta} \left(\sin\theta \frac{\partial}{\partial\theta} \right) + \frac{1}{\sin^2\theta} \frac{\partial^2}{\partial\phi^2} \right].$$

Now, if Y_ℓ^m is an eigenfunction of L^2, with an eigenvalue of $\hbar^2 \ell (\ell + 1)$, then it follows

$$L^2 Y_\ell^m = -\hbar^2 \left[\frac{1}{\sin\theta} \frac{\partial}{\partial\theta} \left(\sin\theta \frac{\partial}{\partial\theta} \right) + \frac{1}{\sin^2\theta} \frac{\partial^2}{\partial\phi^2} \right] Y_\ell^m = \hbar^2 \ell (\ell+1) Y_\ell^m,$$

or simply

$$\boxed{\frac{1}{\sin\theta} \frac{\partial}{\partial\theta} \left(\sin\theta \frac{\partial Y_\ell^m}{\partial\theta} \right) + \frac{1}{\sin^2\theta} \frac{\partial^2 Y_\ell^m}{\partial\phi^2} = -\ell (\ell+1) Y_\ell^m},$$

which is just the angular part of the Schrödinger equation in spherical coordinates. In addition it is an eigenfunction of L_z, with an eigenvalue of $\hbar m$,

$$-i\hbar \frac{\partial}{\partial\phi} Y_\ell^m = \hbar m Y_\ell^m \quad \rightarrow \quad \boxed{\frac{\partial}{\partial\phi} Y_\ell^m = im Y_\ell^m},$$

which is equivalent to the azimuthal equation, which is obtained when we separate variables in the first equation. Thus, we see that we can write the hamiltonian in terms of the angular momentum as:

$$\boxed{\hat{H} = \frac{1}{2mr^2} \left[-\hbar^2 \frac{\partial}{\partial r} \left(r^2 \frac{\partial}{\partial r} \right) + L^2 \right] + V(r)},$$

it is clear that L^2 and L_z (in the chosen basis) commute with the hamiltonian, thus we can represent all stationary states with the two quantum numbers ℓ and m.

Solving the differential equations above yield the eigenfunction in the position basis:

$$Y_\ell^m(\theta, \phi) = \sqrt{\frac{(2\ell+1)}{4\pi} \frac{(\ell-m)!}{(\ell+m)!}} e^{im\phi} P_\ell^m(\cos\theta)$$

where P_ℓ^m is the *associated Legendre function,* defined by

$$P_\ell^m(x) \equiv (-1)^m \left(1 - x^2\right)^{m/2} \left(\frac{d}{dx} \right)^m P_\ell(x), \quad \text{and} \quad P_\ell^{-m}(x) = (-1)^m \frac{(\ell-m)!}{(\ell+m)!} P_\ell^m(x), \quad \text{with } m \geq 0,$$

where $P_\ell(x)$ is the ℓth *Legendre polynomial*, defined by the *Rodrigues formula*:

$$P_\ell(x) \equiv \frac{1}{2^\ell \ell!} \left(\frac{d}{dx} \right)^\ell \left(x^2 - 1\right)^\ell.$$

Some spherical harmonics:

$Y_0^0 = \left(\frac{1}{4\pi}\right)^{1/2}$	$Y_2^{\pm2} = \left(\frac{15}{32\pi}\right)^{1/2} \sin^2\theta e^{\pm 2i\phi}$
$Y_1^0 = \left(\frac{3}{4\pi}\right)^{1/2} \cos\theta$	$Y_3^0 = \left(\frac{7}{16\pi}\right)^{1/2} \left(5\cos^3\theta - 3\cos\theta\right)$
$Y_1^{\pm1} = \mp \left(\frac{3}{8\pi}\right)^{1/2} \sin\theta e^{\pm i\phi}$	$Y_3^{\pm1} = \mp \left(\frac{21}{64\pi}\right)^{1/2} \left(5\cos^2\theta - 1\right) \theta e^{\pm i\phi}$
$Y_2^0 = \left(\frac{5}{16\pi}\right)^{1/2} \left(3\cos^2\theta - 1\right)$	$Y_3^{\pm2} = \left(\frac{105}{32\pi}\right)^{1/2} \sin^2\theta \cos\theta e^{\pm 2i\phi}$
$Y_2^{\pm1} = \mp \left(\frac{15}{8\pi}\right)^{1/2} \sin\theta \cos\theta e^{\pm i\phi}$	$Y_3^{\pm3} = \mp \left(\frac{35}{64\pi}\right)^{1/2} \sin^3\theta e^{\pm 3i\phi}$

Question 2:

Consider a particle moving in a spherically symmetric potential. The particle is prepared at $t = 0$ at a state

$$|\psi\rangle = \frac{A}{r}(z + ix) f(r),$$

where $f(r)$ is normalized such that

$$\int_0^\infty f(r) r^2 dr = 1.$$

1. Is this state an eigenstate of $\hat{L}^2$? What are the possible outcomes for the measurement of $\hat{L}_z$ and what are the probabilities of measuring them?
2. We measure $\hat{L}_x$. What is the probability to measure a value $\hbar$?

Solution:

1. Let us move to spherical coordinates:

   $$|\psi\rangle = A[\cos\theta + i\sin\theta\cos\phi] f(r).$$

   looking at the eigenfunctions of $\hat{L}^2$ we see that

   $$\cos\theta = \left(\frac{4\pi}{3}\right)^{1/2} Y_1^0,$$

   and

   $$\sin\theta\cos\phi = \frac{1}{2}\sin\theta\left(e^{i\phi} + e^{-i\phi}\right) = \frac{1}{2}\left(\frac{8\pi}{3}\right)^{1/2}\left(-Y_1^1 + Y_1^{-1}\right),$$

   thus

   $$|\psi\rangle = \tilde{A}\left(\sqrt{2}Y_1^0 - iY_1^1 + iY_1^{-1}\right) f(r).$$

   Since $f(r)$ depends only on the radial coordinate, it is unaffected by the angular momentum and we may write $|\psi\rangle$ as

   $$\boxed{|\psi\rangle = \tilde{A}\left(\sqrt{2}\,|1,0\rangle - i\,|1,1\rangle + i\,|1,-1\rangle\right) \otimes |f\rangle},$$

   where $\tilde{A}$ can be determined from normalization:

   $$1 = \langle\psi|\psi\rangle = \left|\tilde{A}\right|^2 (2 + 1 + 1) \quad \rightarrow \quad |A| = \frac{1}{2}.$$

   Therefore, when measuring $\hat{L}_z$ we can get

   $$\boxed{P(L_z = \hbar) = \frac{1}{4} \quad P(L_z = 0) = \frac{1}{2} \quad P(L_z = -\hbar) = \frac{1}{4}}.$$

2. We wish to find

   $$P(\psi_x) = |\langle\psi_x|\psi\rangle|^2,$$

   where

   $$L_x|\psi_x\rangle = \hbar|\psi_x\rangle,$$

   thus, we need to find ψ_x. Since ψ_x is a state of the system, then

   $$|\psi_x\rangle = C(c_1|1,0\rangle + c_2|1,1\rangle + c_3|1,-1\rangle) \equiv C(c_1|0\rangle + c_2|1\rangle + c_3|-1\rangle),$$

   where we dropped the common $\ell = 1$ number. Next we recall that

   $$L_x = \frac{1}{2}(L_+ + L_-),$$

and

$$L_\pm \left|\ell, m\right\rangle = \hbar\sqrt{(\ell \mp m)(\ell \pm m + 1)}\left|\ell, m \pm 1\right\rangle,$$

thus

$$\begin{aligned}
L_x \left|\psi_x\right\rangle &= \frac{C}{2}\left(L_+ + L_-\right)\left(c_1\left|0\right\rangle + c_2\left|1\right\rangle + c_3\left|-1\right\rangle\right) \\
&= \frac{C}{2}\left(c_1 L_+\left|0\right\rangle + c_2\cancel{L_+\left|1\right\rangle} + c_3 L_+\left|-1\right\rangle + c_1 L_-\left|0\right\rangle + c_2 L_-\left|1\right\rangle + c_3\cancel{L_-\left|-1\right\rangle}\right) \\
&= \frac{C}{\sqrt{2}}\hbar\left(c_1\left|1\right\rangle + c_1\left|-1\right\rangle + c_2\left|0\right\rangle + c_3\left|0\right\rangle\right).
\end{aligned}$$

Comparing with the equation for $\left|\psi_x\right\rangle$, we find

$$\begin{aligned}
c_1 &= \frac{1}{\sqrt{2}}\left(c_2 + c_3\right) \\
c_2 &= \frac{1}{\sqrt{2}}c_1 \\
c_3 &= \frac{1}{\sqrt{2}}c_1
\end{aligned}$$

hence

$$\left|\psi_x\right\rangle = \tilde{C}\left(\left|0\right\rangle + \frac{1}{\sqrt{2}}\left|1\right\rangle + \frac{1}{\sqrt{2}}\left|-1\right\rangle\right),$$

where $\tilde{C} = 1/\sqrt{2}$ is determined from normalization. Finally

$$P\left(\psi_x\right) = \left|\left\langle\psi_x|\psi\right\rangle\right|^2 = \left|\frac{1}{2\sqrt{2}}\left(\sqrt{2} - \frac{i}{\sqrt{2}} + \frac{i}{\sqrt{2}}\right)\right|^2 = \frac{1}{4}.$$

The same procedure can be carried using matrix representation:
Since $\left|\psi\right\rangle$ is composed of $\ell = 1$ states, we may refer only to the 3×3 L_z matrix, which looks like

$$\left(L_z\right)_{nm} = \left\langle n|\hat{L}_z|m\right\rangle = \hbar m\delta_{nm} \quad \to \quad L_z = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

with the eigenvectors

$$\left|1\right\rangle \to \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}; \quad \left|0\right\rangle \to \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}; \quad \left|-1\right\rangle \to \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

The $L_\pm$ matrices are

$$\left(L_\pm\right)_{nm} = \left\langle n|\hat{L}_\pm|m\right\rangle = \hbar\sqrt{(1 \mp m)(1 \pm m + 1)}\delta_{n,m\pm 1},$$

$$L_+ = \hbar\left[\begin{array}{cc} & \begin{array}{ccc} \left|1\right\rangle & \left|0\right\rangle & \left|-1\right\rangle \end{array} \\ \begin{array}{c} \left\langle 1\right| \\ \left\langle 0\right| \\ \left\langle -1\right| \end{array} & \begin{pmatrix} 0 & \sqrt{2} & 0 \\ 0 & 0 & \sqrt{2} \\ 0 & 0 & 0 \end{pmatrix} \end{array}\right]; \quad L_- = \hbar\left[\begin{array}{cc} & \begin{array}{ccc} \left|1\right\rangle & \left|0\right\rangle & \left|-1\right\rangle \end{array} \\ \begin{array}{c} \left\langle 1\right| \\ \left\langle 0\right| \\ \left\langle -1\right| \end{array} & \begin{pmatrix} 0 & 0 & 0 \\ \sqrt{2} & 0 & 0 \\ 0 & \sqrt{2} & 0 \end{pmatrix} \end{array}\right],$$

thus

$$L_x = \frac{\hbar}{\sqrt{2}}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}.$$

We want to find $\left|\psi_x\right\rangle$ which follows

$$L_x\left|\psi_x\right\rangle = \hbar\left|\psi_x\right\rangle \quad \to \quad \frac{\hbar}{\sqrt{2}}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}\left|\psi_x\right\rangle = \hbar\left|\psi_x\right\rangle \quad \to \quad \left|\psi_x\right\rangle = \frac{1}{2}\begin{pmatrix} 1 \\ \sqrt{2} \\ 1 \end{pmatrix},$$

so that

$$P(\psi_x) = |\langle\psi_x|\psi\rangle|^2 = \left|\frac{1}{4}\begin{pmatrix} 1 & \sqrt{2} & 1 \end{pmatrix}\begin{pmatrix} i \\ \sqrt{2} \\ -i \end{pmatrix}\right|^2 = \frac{1}{4}.$$