Homework 1 - Fourier Transform

Question 1:

Show that $a_n \sin \frac{\pi n x}{L}$ and $b_n \cos \frac{\pi n x}{L}$ form an orthogonal basis in the domain $[-L, L]$. Normalize the basis by finding the appropriate a_n and b_n.

Solution:

Taking the inner product for all combinations we find

$$\begin{aligned}
\left\langle a_m \sin \frac{\pi m x}{L}, a_n \sin \frac{\pi n x}{L} \right\rangle &= \int_{-L}^{L} a_m^* \sin \frac{\pi m x}{L} a_n \sin \frac{\pi n x}{L} dx \\
&= a_m^* a_n \int_{-L}^{L} \frac{1}{2} \left[\cos \left(\frac{\pi x}{L} (m-n) \right) - \cos \left(\frac{\pi x}{L} (m+n) \right) \right] dx \\
\text{if } n \neq m \text{ then: } &= \frac{1}{2} a_m^* a_n \left[\frac{L}{\pi (m-n)} \sin \left(\frac{\pi x}{L} (m-n) \right) - \frac{L}{\pi (m+n)} \sin \left(\frac{\pi x}{L} (m+n) \right) \right]_{-L}^{L}
\end{aligned}$$

which leads to terms proportional to $\sin(\pi (m \pm n)) = 0$.

$$\begin{aligned}
\text{if } n = m \text{ then: } &= a_m^* a_n \int_{-L}^{L} \frac{1}{2} \left[1 - \cos \left(\frac{2\pi n x}{L} \right) \right] dx \\
&= a_m^* a_n \frac{1}{2} \left[x - \cancel{\frac{L}{2\pi n} \sin \left(\frac{2\pi n x}{L} \right)} \right]_{-L}^{L} \\
&= a_m^* a_n L,
\end{aligned}$$

thus the two are orthogonal,

$$\boxed{\left\langle a_m \sin \frac{\pi m x}{L}, a_n \sin \frac{\pi n x}{L} \right\rangle = L a_m^* a_n \delta_{nm}} = L |a_n|^2 .$$

Requiring orthonormality we find

$$\boxed{|a_n| = L^{-1/2}}.$$

Next is

$$\begin{aligned}
\left\langle b_m \cos \frac{\pi m x}{L}, b_n \cos \frac{\pi n x}{L} \right\rangle &= \int_{-L}^{L} b_m^* \cos \frac{\pi m x}{L} b_n \cos \frac{\pi n x}{L} dx \\
&= b_m^* b_n \int_{-L}^{L} \frac{1}{2} \left[\cos \left(\frac{\pi x}{L} (m-n) \right) + \cos \left(\frac{\pi x}{L} (m+n) \right) \right] dx \\
\text{if } n \neq m \text{ then: } &= \frac{1}{2} b_m^* b_n \left[\frac{L}{\pi (m-n)} \sin \left(\frac{\pi x}{L} (m-n) \right) + \frac{L}{\pi (m+n)} \sin \left(\frac{\pi x}{L} (m+n) \right) \right]_{-L}^{L}
\end{aligned}$$

which leads to terms proportional to $\sin(\pi (m \pm n)) = 0$.

$$\begin{aligned}
\text{if } n = m \text{ then: } &= b_n^* b_n \int_{-L}^{L} \frac{1}{2} \left[1 + \cos \left(\frac{2\pi n x}{L} \right) \right] dx \\
&= |b_n|^2 \frac{1}{2} \left[x + \cancel{\frac{L}{2\pi n} \sin \left(\frac{2\pi n x}{L} \right)} \right]_{-L}^{L} \\
&= |b_n|^2 L,
\end{aligned}$$

thus the two are orthogonal,

$$\boxed{\left\langle b_m \cos\frac{\pi m x}{L}, b_n \cos\frac{\pi n x}{L}\right\rangle = L b_m^* b_n \delta_{nm}} = L\left|b_n\right|^2.$$

Requiring orthonormality we find

$$\boxed{|b_n| = L^{-1/2}}.$$

And last is the cross term

$$\begin{aligned}\left\langle a_m \sin\frac{\pi m x}{L}, b_n \cos\frac{\pi n x}{L}\right\rangle &= \int_{-L}^{L} a_m^* \sin\frac{\pi m x}{L} b_n \cos\frac{\pi n x}{L} dx \\ &= a_m^* b_n \int_{-L}^{L} \frac{1}{2}\left[\sin\left(\frac{\pi x}{L}(m+n)\right) + \sin\left(\frac{\pi x}{L}(m-n)\right)\right] dx = 0,\end{aligned}$$

where we used the fact that this is a symmetric integral over a sum of antisymmetric functions and thus it vanishes - indicating that the two sets are orthogonal.

Question 2:

Find the Fourier coefficients for the function $f(x) = 1$ in the domain $[-\pi, \pi]$.

Solution:

Expanding $f(x)$ into Fourier series we have

$$f(x) = \sum_n c_n e^{inx},$$

where

$$c_n = \frac{1}{\sqrt{2\pi}} \int_{-\pi}^{\pi} e^{-inx} dx.$$

If $n = 0$ we find

$$\boxed{c_0 = 1},$$

if $n \neq 0$ then,

$$\boxed{c_n = \frac{1}{in\sqrt{2\pi}}\left(e^{in\pi} - e^{-in\pi}\right) = \sqrt{2\pi}\frac{\sin\pi n}{\pi n} = 0 \quad \forall n \in \text{int}}.$$

Question 3:

Derive the Fourier series for the function $f(x) = x^2$ in the domain $[-\pi, \pi]$. Use this to prove that

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}.$$

Solution:

Expanding $f(x)$ into Fourier series we have

$$f(x) = \frac{1}{\sqrt{2\pi}} \sum_n c_n e^{inx},$$

where

$$c_n = \frac{1}{\sqrt{2\pi}} \int_{-\pi}^{\pi} x^2 e^{-inx} dx.$$

If $n = 0$ we find

$$c_0 = \frac{1}{\sqrt{2\pi}} \int_{-\pi}^{\pi} x^2 dx = \sqrt{2\pi}\frac{\pi^2}{3},$$

if $n \neq 0$ then,

$$
\begin{aligned}
c_n &= -\frac{1}{\sqrt{2\pi}} \frac{d^2}{dn^2} \int_{-\pi}^{\pi} e^{-inx} dx \\
&= \frac{1}{\sqrt{2\pi}} \frac{d^2}{dn^2} \left[\frac{1}{in} \left(e^{-in\pi} - e^{in\pi} \right) \right] \\
&= \frac{1}{\sqrt{2\pi}} \frac{d}{dn} \left[\frac{1}{in^2} \left(e^{in\pi} - e^{-in\pi} \right) - \frac{\pi}{n} \left(e^{in\pi} + e^{-in\pi} \right) \right] \\
&= \frac{1}{\sqrt{2\pi}} \frac{d}{dn} \left[\frac{2}{n^2} \sin(n\pi) - \frac{2\pi}{n} \cos(n\pi) \right] \\
&= \sqrt{\frac{2}{\pi}} \left[\left(\frac{\pi^2}{n} - \frac{2}{n^3} \right) \cancel{\sin(n\pi)} + \frac{2\pi}{n^2} \underbrace{\cos(n\pi)}_{(-1)^n} \right] \\
&= \frac{2\sqrt{2\pi}}{n^2} (-1)^n .
\end{aligned}
$$

Therefore, the Fourier series is

$$
\begin{aligned}
f(x) &= \frac{1}{\sqrt{2\pi}} \sum_n c_n e^{inx} \\
&= \frac{\pi^2}{3} + \sum_{n \neq 0} \frac{2}{n^2} (-1)^n e^{inx} \\
&= \frac{\pi^2}{3} + \sum_{n=1}^{\infty} \frac{2}{n^2} (-1)^n \left(e^{inx} + e^{-inx} \right),
\end{aligned}
$$

or simply

$$
\boxed{f(x) = \frac{\pi^2}{3} + \sum_{n=1}^{\infty} \frac{4}{n^2} (-1)^n \cos(nx)}.
$$

Now let us take its value at $x = \pi$,

$$
\pi^2 = \frac{\pi^2}{3} + \sum_{n=1}^{\infty} \frac{4}{n^2} (-1)^n \underbrace{\cos(n\pi)}_{(-1)^n} = \frac{\pi^2}{3} + \sum_{n=1}^{\infty} \frac{4}{n^2} \quad \rightarrow \quad \boxed{\frac{\pi^2}{6} = \sum_{n=1}^{\infty} \frac{1}{n^2}}.
$$

Question 4:

Prove the following identities for the Dirac delta function:

1. Show that $\delta(x) = \frac{d\Theta(x)}{dx}$, where

$$
\Theta(x) = \begin{cases} 1 & , \text{ for } 0 < x \\ 0 & , \text{ for } 0 \geq x \end{cases}
$$

   is the Heaviside step function.

2. Show that

$$
\int_{-\infty}^{\infty} \frac{d\delta(x - x_0)}{dx} f(x)\, dx = -\frac{df(x)}{dx}\Big|_{x=x_0} = -f'(x_0).
$$

3. If $f(x_0) = 0$ and $f'(x_0) \neq 0$, then

$$
\delta[f(x)] = \frac{\delta(x - x_0)}{|f'(x_0)|}.
$$

Solution:

1. Taking the derivative of the Heaviside step function can be explicitly as the limit

$$\frac{d\Theta(x)}{dx} = \lim_{\varepsilon\to 0} \frac{\Theta(x+\varepsilon) - \Theta(x-\varepsilon)}{2\varepsilon},$$

plugging this expression into the generic integral over a function $f(x)$ we find

$$\begin{aligned}
\int_{-\infty}^{\infty} f(x) \frac{d\Theta(x)}{dx} dx &= \lim_{\varepsilon\to 0} \int_{-\infty}^{\infty} f(x) \frac{\Theta(x+\varepsilon) - \Theta(x-\varepsilon)}{2\varepsilon} dx \\
&= \lim_{\varepsilon\to 0} \left[\int_{-\varepsilon}^{\infty} \frac{f(x)}{2\varepsilon} dx - \int_{\varepsilon}^{\infty} \frac{f(x)}{2\varepsilon} dx \right] \\
&= \lim_{\varepsilon\to 0} \int_{-\varepsilon}^{\varepsilon} \frac{f(x)}{2\varepsilon} dx,
\end{aligned}$$

since $\int_{-\varepsilon}^{\varepsilon} f(x)\, dx = 2\varepsilon f(0)$, on an infinitesimal interval (i.e. the value of the function times the interval), we get

$$\boxed{\int_{-\infty}^{\infty} f(x) \frac{d\Theta(x)}{dx} dx = f(0)},$$

proving that $d\Theta(x)/dx$ operates just as $\delta(x)$ does.

2. Starting from the left side

$$\begin{aligned}
\int_{-\infty}^{\infty} \frac{d\delta(x-x_0)}{dx} f(x)\, dx &= \int_{-\infty}^{\infty} \frac{d}{dx} \left[\delta(x-x_0) f(x)\right] dx - \int_{-\infty}^{\infty} \delta(x-x_0) \frac{df(x)}{dx} dx \\
&= \cancel{\left[\delta(x-x_0) f(x)\right]_{-\infty}^{\infty}} - \left.\frac{df(x)}{dx}\right|_{x=x_0},
\end{aligned}$$

where we used integration by parts and the fact that $\delta(x-x_0)$ vanishes anywhere but at $x = x_0$, for the boundary term, and the definition of $\delta(x-x_0)$ for the second integral term, which yields

$$\boxed{\int_{-\infty}^{\infty} \frac{d\delta(x-x_0)}{dx} f(x)\, dx = -\left.\frac{df(x)}{dx}\right|_{x=x_0} = -f'(x_0)}.$$

3. Rearranging the equation into

$$|f'(x_0)|\, \delta[f(x)] = \delta(x-x_0),$$

and defining $y = f(x)$ or $x = f^{-1}(y)$, hence $f(x_0) \equiv y_0 = 0$ or $f^{-1}(0) = x_0$, we have

$$dy = f'(x)\, dx \quad \to \quad dx = \frac{dy}{|f'[f^{-1}(y)]|}.$$

Now let us show that the left hand side of the equation above operates as $\delta(x-x_0)$ on some function $g(x)$,

$$\begin{aligned}
\int_{-\infty}^{\infty} |f'(x_0)|\, \delta[f(x)]\, g(x)\, dx &= |f'(x_0)| \int_{-\infty}^{\infty} \delta(y)\, g\left(f^{-1}(y)\right) \frac{dy}{|f'[f^{-1}(y)]|} \\
&= |f'(x_0)| \frac{g\left(f^{-1}(0)\right)}{|f'[f^{-1}(0)]|} \\
&= |f'(x_0)| \frac{g(x_0)}{|f'(x_0)|} \\
&= g(x_0).
\end{aligned}$$

This proves that

$$\boxed{|f'(x_0)|\, \delta[f(x)] = \delta(x-x_0)}.$$

Question 5:

Find the Fourier transform of the Gaussian $g(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{x^2}{2\sigma^2}}$, and show that $\Delta x \Delta k = 1$.

Solution:

Taking the Fourier transform of $g(x)$ yields

$$\begin{aligned}\tilde{g}(k) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} g(x) e^{-ikx} dx \\ &= \frac{1}{\sqrt{2\pi}} \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{x^2}{2\sigma^2}} e^{-ikx} dx.\end{aligned}$$

Looking at the power of the exponent we can write

$$-\frac{1}{2\sigma^2}\left(x^2 + 2i\sigma^2 kx\right) = -\frac{1}{2\sigma^2}\left(x^2 + 2i\sigma^2 kx - \sigma^4 k^2\right) - \frac{\sigma^2 k^2}{2},$$

which leads to

$$\tilde{g}(k) = \frac{e^{-\sigma^2 k^2/2}}{2\pi\sigma} \underbrace{\int_{-\infty}^{\infty} e^{-\frac{u^2}{2\sigma^2}} du}_{I},$$

where we defined $u = x + i\sigma^2 k$. we are left with a simple Gaussian integral I which can be solved as follows

$$I^2 = \left(\int_{-\infty}^{\infty} e^{-\frac{x^2}{2\sigma^2}} dx\right)^2 = \int_{-\infty}^{\infty} e^{-\frac{x^2}{2\sigma^2}} dx \int_{-\infty}^{\infty} e^{-\frac{y^2}{2\sigma^2}} dy = \int_{-\infty}^{\infty} e^{-\frac{x^2+y^2}{2\sigma^2}} dxdy$$

moving to polar coordinates

$$I^2 = \int_0^{2\pi}\int_0^{\infty} e^{-\frac{r^2}{2\sigma^2}} rdrd\varphi = \pi \int_0^{\infty} e^{-\frac{\xi}{2\sigma^2}} d\xi = 2\pi\sigma^2,$$

where we defined $\xi = r^2$ hence $d\xi = 2rdr$, which gives $I = \sqrt{2\pi\sigma^2}$.
Therefore

$$\boxed{\tilde{g}(k) = \frac{e^{-\sigma^2 k^2/2}}{\sqrt{2\pi}}}.$$

Calculating the square root of the variance $\Delta x \equiv \sqrt{\langle x^2\rangle - \langle x\rangle^2}$ of both results we see that

$$\langle x\rangle = \int_{-\infty}^{\infty} x \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{x^2}{2\sigma^2}} dx = 0,$$

where the integral vanishes as it is symmetric and the integrand is anti-symmetric.

$$\left\langle x^2\right\rangle = \int_{-\infty}^{\infty} x^2 \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{x^2}{2\sigma^2}} dx$$

defining $\alpha = \frac{1}{2\sigma^2}$ we can write

$$\begin{aligned}\left\langle x^2\right\rangle &= -\frac{1}{\sigma\sqrt{2\pi}} \frac{d}{d\alpha} \int_{-\infty}^{\infty} e^{-\alpha x^2} dx \\ &= -\frac{1}{\sigma\sqrt{2\pi}} \frac{d}{d\alpha} \sqrt{\pi/\alpha} \\ &= \frac{1}{\sigma 2\sqrt{2}} \left(2\sigma^2\right)^{3/2} \\ &= \sigma^2.\end{aligned}$$

Therefore

$$\Delta x = \sigma.$$

A similar calculation for Δk yields

$$\Delta k = \frac{1}{\sigma},$$

therefore

$$\boxed{\Delta x \Delta k = 1}.$$