Homework 3 - Wave function

Question 1:

Consider a free particle moving in one dimension. At time $t = 0$ its wave function is

$$\psi(x) = N\left[e^{-\frac{\alpha}{2}(x+x_0)^2} + e^{-\frac{\alpha}{2}(x-x_0)^2}\right],$$

where α and x_0 are real parameters such that $\{\alpha, x_0\} > 0$.

1. Compute the normalization factor $|N|$ and the momentum wave function $\tilde{\psi}(k)$.
2. Find the time dependent wave function $\Psi(x,t)$ at any time $t > 0$. *Hint*: Use the notation $z = 1 + i\left(\frac{\hbar\alpha t}{m}\right)$.
3. Write down the position probability density and discuss the physical interpretation of each term.
4. Obtain an expression for the probability current, which is defined as

$$J(x,t) = \frac{i\hbar}{2m}\left(\Psi\frac{\partial\Psi^*}{\partial x} - \Psi^*\frac{\partial\Psi}{\partial x}\right).$$

Solution:

1. Using the normalization condition

$$\int_{-\infty}^{\infty} |\psi|^2 \, dx = 1,$$

gives us

$$\begin{aligned}
\int_{-\infty}^{\infty} |\psi|^2 \, dx &= |N|^2 \int_{-\infty}^{\infty} \left[e^{-\frac{\alpha}{2}(x+x_0)^2} + e^{-\frac{\alpha}{2}(x-x_0)^2}\right]^2 dx \\
&= |N|^2 \int_{-\infty}^{\infty} \left[e^{-\alpha(x+x_0)^2} + 2e^{-\alpha\left(x^2+x_0^2\right)} + e^{-\alpha(x-x_0)^2}\right] dx \\
&= |N|^2 \sqrt{\frac{\pi}{\alpha}}\left[1 + 2e^{-\alpha x_0^2} + 1\right],
\end{aligned}$$

thus

$$\boxed{|N| = \frac{1}{\sqrt{2}}\left(\frac{\alpha}{\pi}\right)^{1/4}\left(1 + \exp\left[-\alpha x_0^2\right]\right)^{-1/2}}.$$

Whereas the Fourier transform is

$$\begin{aligned}
\tilde{\psi}(k) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \psi(x)\, e^{-ikx} dx \\
&= \frac{N}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \left[e^{-\frac{\alpha}{2}(x+x_0)^2} + e^{-\frac{\alpha}{2}(x-x_0)^2}\right] e^{-ikx} dx
\end{aligned}$$

completing the square in the powers reads

$$\begin{aligned}
-\frac{\alpha}{2}(x \pm x_0)^2 - ikx &= -\frac{\alpha}{2}\left(x^2 + 2\left(ik/\alpha \pm x_0\right)x + x_0^2\right) \\
&= -\frac{\alpha}{2}\left(x^2 + 2\left(ik/\alpha \pm x_0\right)x + \left(ik/\alpha \pm x_0\right)^2\right) + \frac{\alpha}{2}\left(ik/\alpha \pm x_0\right)^2 - \frac{\alpha}{2}x_0^2 \\
&= -\frac{\alpha}{2}\left(x + ik/\alpha \pm x_0\right)^2 - \frac{1}{2\alpha}\left(k^2 \mp 2i\alpha x_0 k - \alpha^2 x_0^2\right) - \frac{\alpha}{2}x_0^2 \\
&= -\frac{\alpha}{2}\left(x + ik/\alpha \pm x_0\right)^2 - \frac{k^2}{2\alpha} \pm i x_0 k
\end{aligned}$$

thus

$$\begin{aligned}\tilde{\psi}(k) &= \frac{N}{\sqrt{2\pi}} e^{-\frac{k^2}{2\alpha}} \int_{-\infty}^{\infty} \left[e^{-\frac{\alpha}{2}(x+ik/\alpha+x_0)^2} e^{ix_0k} + e^{-\frac{\alpha}{2}(x+ik/\alpha-x_0)^2} e^{-ix_0k} \right] dx \\ &= \frac{N}{\sqrt{2\pi}} e^{-\frac{k^2}{2\alpha}} \sqrt{\frac{2\pi}{\alpha}} \left[e^{ix_0k} + e^{-ix_0k} \right] \\ &= \frac{2N}{\sqrt{\alpha}} e^{-\frac{k^2}{2\alpha}} \cos x_0 k,\end{aligned}$$

plugging in the normalization factor we get

$$\boxed{\tilde{\psi}(k) = \frac{\sqrt{2}}{(\pi\alpha)^{1/4}} \frac{e^{-\frac{k^2}{2\alpha}}}{\sqrt{1+\exp\left[-\alpha x_0^2\right]}} \cos x_0 k}.$$

2. Taking the inverse Fourier transform, considering the time evolution of a free particle, i.e. $\omega = \hbar k^2/2m$, we have

$$\begin{aligned}\Psi(x,t) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \tilde{\psi}(k)\, e^{i(kx-\omega t)} dk \\ &= \frac{2N}{\sqrt{2\pi\alpha}} \int_{-\infty}^{\infty} e^{-\frac{k^2}{2\alpha}} \cos x_0 k e^{i\left(kx - \frac{\hbar t}{2m} k^2\right)} dk \\ &= \frac{N}{\sqrt{2\pi\alpha}} \int_{-\infty}^{\infty} e^{-\frac{k^2}{2\alpha}\left(1+i\frac{\hbar t\alpha}{m}\right)+ikx} \left(e^{ikx_0} + e^{-ikx_0} \right) dk \\ &= \frac{N}{\sqrt{2\pi\alpha}} \int_{-\infty}^{\infty} \left(e^{-\frac{z}{2\alpha}\left(k^2 - 2i\frac{\alpha k}{z}(x+x_0)\right)} + e^{-\frac{z}{2\alpha}\left(k^2 - 2i\frac{\alpha k}{z}(x-x_0)\right)} \right) dk,\end{aligned}$$

where $z(t) \equiv 1 + i\frac{\hbar\alpha t}{m}$. Looking at the powers

$$\begin{aligned}-\frac{z}{2\alpha}\left(k^2 - 2i\frac{\alpha k}{z}(x \pm x_0)\right) &= -\frac{z}{2\alpha}\left(k^2 - 2i\frac{\alpha}{z}(x\pm x_0)k + \frac{(i\alpha)^2}{z^2}(x\pm x_0)^2\right) - \frac{\alpha}{2z}(x\pm x_0)^2 \\ &= -\frac{z}{2\alpha}\left(k^2 - i\frac{\alpha}{z}(x\pm x_0)\right)^2 - \frac{\alpha}{2z}(x\pm x_0)^2,\end{aligned}$$

thus

$$\Psi(x,t) = \frac{N}{\sqrt{2\pi\alpha}} \sqrt{\frac{2\pi\alpha}{z}} \left(e^{-\frac{\alpha}{2z}(x+x_0)^2} + e^{-\frac{\alpha}{2z}(x-x_0)^2} \right)$$

$$\boxed{\Psi(x,t) = \frac{N}{\sqrt{z(t)}} \left(e^{-\frac{\alpha}{2z(t)}(x+x_0)^2} + e^{-\frac{\alpha}{2z(t)}(x-x_0)^2} \right)},$$

which are the same initial Gaussians we started with, evolved in time.

3. The probability density is

$$\left(e^{-\frac{\alpha}{2z}(x+x_0)^2} + e^{-\frac{\alpha}{2z}(x-x_0)^2} \right) \left(e^{-\frac{\alpha}{2z^*}(x+x_0)^2} + e^{-\frac{\alpha}{2z^*}(x-x_0)^2} \right) =$$

$$\begin{aligned}\rho(x,t) &= \Psi^*\Psi \\ &= \frac{N^2}{\sqrt{|z|^2}} \left(e^{-\frac{\alpha}{2z}(x+x_0)^2} + e^{-\frac{\alpha}{2z}(x-x_0)^2} \right) \left(e^{-\frac{\alpha}{2z^*}(x+x_0)^2} + e^{-\frac{\alpha}{2z^*}(x-x_0)^2} \right) \\ &= \frac{N^2}{\sqrt{|z|^2}} \Bigg(e^{-\frac{\alpha \mathrm{Re}[z]}{|z|^2}(x+x_0)^2} + e^{-\frac{\alpha \mathrm{Re}[z]}{|z|^2}(x-x_0)^2} \\ &\qquad + e^{-\frac{\alpha}{|z|^2}\left(\mathrm{Re}[z]x^2 + \mathrm{Re}[z]x_0^2 + i\mathrm{Im}[z]2x_0x\right)} + e^{-\frac{\alpha}{|z|^2}\left(\mathrm{Re}[z]x^2 + \mathrm{Re}[z]x_0^2 - i\mathrm{Im}[z]2x_0x\right)} \Bigg),\end{aligned}$$

Recalling that $\text{Re}[z] = 1$ and $\text{Im}[z] = \hbar\alpha t/m$ we have

$$\rho(x,t) = \frac{N^2}{\sqrt{|z|^2}}\left[e^{-\frac{\alpha}{|z|^2}(x+x_0)^2} + e^{-\frac{\alpha}{|z|^2}(x-x_0)^2} + e^{-\frac{\alpha}{|z|^2}(x^2+x_0^2)}\left(e^{\frac{i2\hbar t x_0\alpha^2}{m|z|^2}x} + e^{-\frac{i2\hbar t x_0\alpha^2}{m|z|^2}x}\right)\right]$$

$$\boxed{\rho(x,t) = \frac{N^2}{\sqrt{|z|^2}}\left[\underbrace{e^{-\frac{\alpha}{|z|^2}(x+x_0)^2} + e^{-\frac{\alpha}{|z|^2}(x-x_0)^2}}_{\text{original Gaussians}} + \underbrace{2e^{-\frac{\alpha}{|z|^2}(x^2+x_0^2)}\cos\left(2\frac{\hbar t}{m}\frac{\alpha^2 x_0 x}{|z|^2}\right)}_{\text{Interference term}}\right]}.$$

4. Note that

$$J(x,t) = \frac{i\hbar}{2m}\left(\Psi\frac{\partial\Psi^*}{\partial x} - \Psi^*\frac{\partial\Psi}{\partial x}\right) = -\frac{\hbar}{m}\text{Im}\left[\Psi\frac{\partial\Psi^*}{\partial x}\right].$$

Taking the spatial derivative of Ψ^*,

$$\begin{aligned}
\Psi\frac{\partial\Psi^*}{\partial x} &= -\frac{\alpha}{z^*}\frac{N^2}{\sqrt{|z|^2}}\left(e^{-\frac{\alpha}{2z}(x+x_0)^2} + e^{-\frac{\alpha}{2z}(x-x_0)^2}\right)\left((x+x_0)\,e^{-\frac{\alpha}{2z^*}(x+x_0)^2} + (x-x_0)\,e^{-\frac{\alpha}{2z^*}(x-x_0)^2}\right) \\
&= -\frac{\alpha}{z^*}\frac{N^2}{\sqrt{|z|^2}}\Bigg[(x+x_0)\,e^{-\frac{\alpha}{|z|^2}(x+x_0)^2} + (x-x_0)\,e^{-\frac{\alpha}{|z|^2}(x-x_0)^2} \\
&\qquad + 2e^{-\frac{\alpha}{|z|^2}(x^2+x_0^2)}\left(x\cos\left(2\frac{\hbar t}{m}\frac{\alpha^2 x_0 x}{|z|^2}\right) - ix_0\sin\left(2\frac{\hbar t}{m}\frac{\alpha^2 x_0 x}{|z|^2}\right)\right)\Bigg], \\
&= -\frac{\left(1+i\frac{\hbar\alpha t}{m}\right)\alpha N^2}{|z|^3}\Bigg[(x+x_0)\,e^{-\frac{\alpha}{|z|^2}(x+x_0)^2} + (x-x_0)\,e^{-\frac{\alpha}{|z|^2}(x-x_0)^2} \\
&\qquad + 2e^{-\frac{\alpha}{|z|^2}(x^2+x_0^2)}\left(x\cos\left(2\frac{\hbar t}{m}\frac{\alpha^2 x_0 x}{|z|^2}\right) - ix_0\sin\left(2\frac{\hbar t}{m}\frac{\alpha^2 x_0 x}{|z|^2}\right)\right)\Bigg].
\end{aligned}$$

Therefore

$$\begin{aligned}
J(x,t) = \frac{\alpha N^2}{|z|^3}\frac{\hbar}{m}\Bigg[&\frac{\hbar\alpha t}{m}\left((x+x_0)\,e^{-\frac{\alpha}{|z|^2}(x+x_0)^2} + (x-x_0)\,e^{-\frac{\alpha}{|z|^2}(x-x_0)^2}\right) \\
&e^{-\frac{\alpha}{|z|^2}(x^2+x_0^2)}\left(\frac{\hbar\alpha t}{m}x\cos\left(2\frac{\hbar t}{m}\frac{\alpha^2 x_0 x}{|z|^2}\right) - 2x_0\sin\left(2\frac{\hbar t}{m}\frac{\alpha^2 x_0 x}{|z|^2}\right)\right)\Bigg].
\end{aligned}$$

Question 2:

Consider a free particle that is initially (at $t = 0$) extremely well localized at the origin (i.e. at $x = 0$) and has a Gaussian wave function,

$$\psi(x) = \left(\frac{\alpha}{\pi}\right)^{1/4} e^{-\frac{\alpha x^2}{2}} e^{ik_0 x},$$

where α is very large ($\alpha \gg 1$), and k_0 is a real constant.

1. Write down position probability density $\rho(x,0)$ and show that

$$\lim_{\alpha\to\infty}\rho(x,0) = \delta(x).$$

2. Keep α finite and calculate $\tilde{\Psi}(x,t)$ at times $t > 0$ and the k-space probability density $\tilde{\rho}(k,t)$. What is the most momentum of the particle?

3. Calculate $\Psi(x,t)$ at times $t > 0$ and the position probability density $\rho(x,t)$. What is the most probable position as a function of time? What happens at the limit $t \to \infty$?

4. Calculate the expectation values $\langle p(t)\rangle$ and $\langle x(t)\rangle$ and show that they follow the classical laws of motion for a free particle, that is

$$\langle p\rangle_t = m\frac{d\langle x\rangle_t}{dt} \quad \text{and} \quad \frac{d\langle p(t)\rangle}{dt} = m\frac{d^2\langle x\rangle_t}{dt^2} = 0.$$

5. Calculate the expectation values $\langle p^2(t)\rangle$ and $\langle x^2(t)\rangle$ and check consistency with the position-momentum uncertainty principle, i.e. $(\Delta x(t))^2(\Delta p(t))^2 \geq \hbar^2/4$.

6. What is the physical meaning of the quantity $\tau = \frac{\Delta x(t)}{|d\langle x(t)\rangle/dt|}$ (beyond the immediate conclusion from units). Show that it is consistent with the time-energy uncertainty principle $\tau\Delta E \geq \hbar/2$.

Solution:

1. The position probability is

$$\rho(x) = \psi^*\psi = \sqrt{\frac{\alpha}{\pi}}e^{-\alpha x^2}.$$

Taking the integral over a multiplication of a function $f(x)$ and the limit

$$\int_{-\infty}^{\infty} f(x)\lim_{\alpha\to\infty}\rho(x)\,dx = \int_{-\infty}^{\infty}\lim_{\alpha\to\infty} f(x)\sqrt{\frac{\alpha}{\pi}}e^{-\alpha x^2}dx,$$

changing variables to $u = \sqrt{\alpha}x$, thus $dx = du/\sqrt{\alpha}$, we get

$$\begin{aligned}\int_{-\infty}^{\infty} f(x)\lim_{\alpha\to\infty}\rho(x)\,dx &= \int_{-\infty}^{\infty}\lim_{\alpha\to\infty} f\left(\frac{u}{\sqrt{\alpha}}\right)\frac{1}{\sqrt{\pi}}e^{-u^2}du \\ &= \int_{-\infty}^{\infty} f(0)\frac{1}{\sqrt{\pi}}e^{-u^2}du \\ &= f(0),\end{aligned}$$

which proves that

$$\boxed{\lim_{\alpha\to\infty}\rho(x,0) = \delta(x)}.$$

2. The Fourier transform is

$$\begin{aligned}\tilde{\psi}(k) &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\psi(x)e^{-ikx}dx \\ &= \frac{1}{\sqrt{2\pi}}\left(\frac{\alpha}{\pi}\right)^{1/4}\int_{-\infty}^{\infty}e^{-\frac{\alpha x^2}{2}}e^{-i(k-k_0)x}dx \\ &= \frac{1}{\sqrt{2\pi}}\left(\frac{\alpha}{\pi}\right)^{1/4}\int_{-\infty}^{\infty}e^{-\frac{\alpha}{2}\left(x+i\frac{(k-k_0)}{\alpha}\right)^2-\frac{(k-k_0)^2}{2\alpha}}dx \\ &= \frac{1}{\sqrt{2\pi}}\left(\frac{\alpha}{\pi}\right)^{1/4}e^{-\frac{(k-k_0)^2}{2\alpha}}\sqrt{\frac{2\pi}{\alpha}},\end{aligned}$$

thus

$$\boxed{\tilde{\Psi}(k,t) = \frac{e^{-\frac{(k-k_0)^2}{2\alpha}}}{(\alpha\pi)^{1/4}}e^{-i\frac{\hbar k^2}{2m}t}}.$$

The k-space probability density is

$$\boxed{\tilde{\rho}(k,t) = \tilde{\Psi}(k,t)\Psi(k,t) = \frac{e^{-\frac{(k-k_0)^2}{\alpha}}}{\sqrt{\alpha\pi}}},$$

therefore, the most probable momentum is $\boxed{p_{\max} = \hbar k_{\max} = \hbar k_0}$.

3. Taking the inverse Fourier transform we find

$$\begin{aligned}
\Psi(x,t) &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\Psi(k,t)\,e^{ikx}dk \\
&= \frac{1}{(\alpha\pi)^{1/4}}\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-\frac{(k-k_0)^2}{2\alpha}}e^{-i\frac{\hbar k^2}{2m}t}e^{ikx}dk \\
&= \frac{1}{(\alpha\pi)^{1/4}}\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-\frac{1+i\hbar t\alpha/m}{2\alpha}\left(k^2-2\left(\frac{k_0+i\alpha x}{1+i\hbar t\alpha/m}\right)k\right)-\frac{1}{2\alpha}k_0^2}dk \\
&= \frac{1}{(\alpha\pi)^{1/4}}\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-\frac{1+i\hbar t\alpha/m}{2\alpha}\left(k-\frac{k_0+i\alpha x}{1+i\hbar t\alpha/m}\right)^2+\frac{1}{2\alpha}(1+i\hbar t\alpha/m)\left(\frac{k_0+i\alpha x}{1+i\hbar t\alpha/m}\right)^2-\frac{1}{2\alpha}k_0^2}dk \\
&= \frac{1}{(\alpha\pi)^{1/4}}\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-\frac{1+i\hbar t\alpha/m}{2\alpha}\left(k-\frac{k_0+i\alpha x}{1+i\hbar t\alpha/m}\right)^2-\frac{\alpha}{2}\frac{(x-ik_0/\alpha)^2}{1+i\hbar t\alpha/m}-\frac{1}{2\alpha}k_0^2}dk \\
&= \frac{1}{(\alpha\pi)^{1/4}}\sqrt{\frac{\alpha}{1+i\hbar t\alpha/m}}e^{-\frac{\alpha}{2}\frac{(x-ik_0/\alpha)^2}{1+i\hbar t\alpha/m}-\frac{1}{2\alpha}k_0^2} \\
&= \frac{1}{(\alpha\pi)^{1/4}}\sqrt{\frac{\alpha}{1+i\hbar t\alpha/m}}e^{-\frac{\alpha}{2}\frac{x^2-k_0^2/\alpha^2-2k_0x\hbar t/m-2ik_0x/\alpha-\left(x^2-k_0^2/\alpha^2\right)i\hbar t\alpha/m}{1+\hbar^2t^2\alpha^2/m^2}-\frac{1}{2\alpha}k_0^2}.
\end{aligned}$$

Thus, the time dependent probability density is

$$\rho(x,t) = \Psi^*\Psi = \sqrt{\frac{\alpha}{\pi}}\sqrt{\frac{1}{1+\hbar^2t^2\alpha^2/m^2}}e^{-\alpha\frac{x^2-k_0^2/\alpha^2-2k_0x\hbar t/m}{1+\hbar^2t^2\alpha^2/m^2}-\frac{1}{\alpha}k_0^2},$$

looking at the power we have

$$\begin{aligned}
-\alpha\frac{x^2-k_0^2/\alpha^2-2k_0x\hbar t/m}{1+\hbar^2t^2\alpha^2/m^2}-\frac{1}{2\alpha}k_0^2 &= -\alpha\frac{(x-k_0\hbar t/m)^2-\frac{k_0^2}{\alpha^2}\left(1+\hbar^2t^2\alpha^2/m^2\right)}{1+\hbar^2t^2\alpha^2/m^2}-\frac{1}{\alpha}k_0^2 \\
&= -\alpha\frac{(x-k_0\hbar t/m)^2}{1+\hbar^2t^2\alpha^2/m^2},
\end{aligned}$$

so that

$$\boxed{\rho(x,t) = \sqrt{\frac{\alpha}{\pi}}\sqrt{\frac{1}{1+\hbar^2t^2\alpha^2/m^2}}e^{-\alpha\frac{(x-k_0\hbar t/m)^2}{1+\hbar^2t^2\alpha^2/m^2}}}.$$

The most probable position is $\boxed{x_{\max} = k_0\hbar t/m}$. When taking the limit $t\to\infty$ we find

$$\lim_{t\to\infty}\rho(x,t) = \sqrt{\frac{1}{\alpha\pi}\frac{m}{\hbar t}}e^{-\frac{k_0^2}{\alpha}},$$

i.e. the distribution does not depend on the position after a long time, it goes to zero at each point as well (since it smears over the entire space).

4. The position expectation value is

$$\begin{aligned}
\langle x(t)\rangle &= \int_{-\infty}^{\infty}\Psi^* x\Psi dx \\
&= \sqrt{\frac{\alpha}{\pi}}\sqrt{\frac{1}{1+\hbar^2t^2\alpha^2/m^2}}\int_{-\infty}^{\infty}xe^{-\alpha\frac{(x-k_0\hbar t/m)^2}{1+\hbar^2t^2\alpha^2/m^2}}dx \\
&= \sqrt{\frac{\alpha}{\pi}}\sqrt{\frac{1}{1+\hbar^2t^2\alpha^2/m^2}}\int_{-\infty}^{\infty}(u+k_0\hbar t/m)\,e^{-\alpha\frac{u^2}{1+\hbar^2t^2\alpha^2/m^2}}du,
\end{aligned}$$

for which the first term vanishes since we symmetrically integrate an odd function, and we are left with

$$\boxed{\langle x(t)\rangle = \frac{\hbar k_0}{m}t}.$$

While the momentum expectation value is

$$\langle p(t)\rangle = \hbar\int_{-\infty}^{\infty}\tilde{\Psi}^* k\tilde{\Psi}dk = \frac{\hbar}{\sqrt{\alpha\pi}}\int_{-\infty}^{\infty} ke^{-\frac{(k-k_0)^2}{\alpha}}dk = \frac{\hbar}{\sqrt{\alpha\pi}}\int_{-\infty}^{\infty}(s+k_0)\, e^{-\frac{s^2}{\alpha}}ds,$$

just like in the position calculation we find

$$\boxed{\langle p(t)\rangle = \hbar k_0}.$$

It is straightforward to see that these follow the classical laws of motion,

$$\langle p\rangle_t = m\frac{d\langle x\rangle_t}{dt} \quad \text{and} \quad \frac{d\langle p(t)\rangle}{dt} = m\frac{d^2\langle x\rangle_t}{dt^2} = 0.$$

5. The second moment for the position distribution is

$$\begin{aligned}\left\langle x^2(t)\right\rangle &= \int_{-\infty}^{\infty}\Psi^* x^2\Psi dx \\ &= \sqrt{\frac{\alpha}{\pi}}\sqrt{\frac{1}{1+\hbar^2t^2\alpha^2/m^2}}\int_{-\infty}^{\infty}x^2 e^{-\alpha\frac{(x-k_0\hbar t/m)^2}{1+\hbar^2t^2\alpha^2/m^2}}dx \\ &= \sqrt{\frac{\alpha}{\pi}}\sqrt{\frac{1}{1+\hbar^2t^2\alpha^2/m^2}}\int_{-\infty}^{\infty}\left(u^2 + \cancel{2\frac{k_0\hbar t}{m}u} + \frac{k_0^2\hbar^2t^2}{m^2}\right)e^{-\alpha\frac{u^2}{1+\hbar^2t^2\alpha^2/m^2}}du,\end{aligned}$$

evaluating the nonzero integrated terms yields

$$\left\langle x^2(t)\right\rangle = \left(\frac{\hbar k_0}{m}t\right)^2 + \frac{1+\hbar^2t^2\alpha^2/m^2}{2\alpha},$$

thus

$$\boxed{(\Delta x(t))^2 = \left\langle x^2(t)\right\rangle - \langle x(t)\rangle^2 = \frac{1+\hbar^2t^2\alpha^2/m^2}{2\alpha}}.$$

While the second moment for the momentum distribution is

$$\left\langle p^2(t)\right\rangle = \hbar^2\int_{-\infty}^{\infty}\tilde{\Psi}^* k^2\tilde{\Psi}dk = \frac{\hbar^2}{\sqrt{\alpha\pi}}\int_{-\infty}^{\infty}k^2 e^{-\frac{(k-k_0)^2}{\alpha}}dk = \frac{\hbar^2}{\sqrt{\alpha\pi}}\int_{-\infty}^{\infty}\left(s^2+2sk_0+k_0^2\right)e^{-\frac{s^2}{\alpha}}dk,$$

evaluating the nonzero integrated terms yields

$$\left\langle p^2(t)\right\rangle = \hbar^2k_0^2 + \frac{\hbar^2\alpha}{2},$$

thus

$$\boxed{(\Delta p(t))^2 = \left\langle p^2(t)\right\rangle - \langle p(t)\rangle^2 = \frac{\hbar^2\alpha}{2}}.$$

So the position-momentum uncertainty holds,

$$[\Delta x(t)\,\Delta p(t)]^2 = \frac{\hbar^2}{4}\left(1+\hbar^2t^2\alpha^2/m^2\right) > \frac{\hbar^2}{4}.$$

6. The quantity

$$\tau = \frac{\Delta x(t)}{|d\langle x(t)\rangle/dt|} = \frac{m}{\hbar k_0}\sqrt{\frac{1+\hbar^2t^2\alpha^2/m^2}{2\alpha}}$$

quantifies the time it takes the expectation value to shift past its spread, i.e. the typical spreading time. It is minimal at $t=0$ when its value is $\tau(t=0) = \frac{m}{\hbar k_0\sqrt{2\alpha}}$.