Homework 4 - Wave function

Question 1:

Consider a system with a real Hamiltonian that occupies a stationary state having a real wave function both at time$t = 0$ and $t = t_1$. Meaning, the wave function satisfies:

$$\Psi^* (x, 0) = \Psi (x, 0) \quad \text{and} \quad \Psi^* (x, t_1) = \Psi (x, t_1) .$$

1. Show that such a system must be periodic, i.e. that there exists a period T such that

$$\Psi (x, t) = \Psi (x, t + T) \quad \forall t.$$

2. Calculate the period T.
3. Show that for such a system, the energy eigenvalues must be integer multiples of $2\pi/T$.

Hint: Assume the state $\Psi (x, t)$ has some defined energy E, then in (3) show it must obey the given condition.
Solution:

1. The time dependent wave function at time t_1 is

$$\Psi (x, t_1) = e^{-i\frac{E}{\hbar}t_1} \psi (x) ,$$

which, for a real Hamiltonian (since a real Hamiltonian $\hat{H}$ has a real eigenvalue E), has the complex conjugate

$$\Psi^* (x, t_1) = e^{i\frac{E}{\hbar}t_1} \psi^* (x) = e^{i\frac{E}{\hbar}t_1} \psi (x) ,$$

where we used $\psi (x) = \Psi (x, 0)$. Thus

$$e^{i\frac{E}{\hbar}t_1} \psi (x) = e^{-i\frac{E}{\hbar}t_1} \psi (x) \quad \rightarrow \quad e^{i\frac{E}{\hbar}2t_1} = 1.$$

For a general time t the wave function reads

$$\Psi (x, t) = e^{-i\frac{E}{\hbar}t} \psi (x) = e^{-i\frac{E}{\hbar}t} e^{-i\frac{E}{\hbar}2t_1} \psi (x) = e^{-i\frac{E}{\hbar}(t+2t_1)} \psi (x) = \Psi (x, t + 2t_1) ,$$

thus

$$\boxed{\Psi (x, t) = \Psi (x, t + T) \quad \text{where} \quad T = 2t_1}.$$

2. As shown in (1) the period is $T = 2t_1$.
3. Using the equation we found for T,

$$e^{i\frac{E}{\hbar}T} = 1 \quad \rightarrow \quad \frac{E}{\hbar}T = 2\pi n \quad \rightarrow \quad \boxed{E = \frac{2\pi\hbar}{T}n, \quad n \in \mathbb{Z}}.$$

Question 2:

Consider a normalized wave function $\psi_0(x)$. Assume that the system is in a state described by the wave function

$$\psi(x) = C_1\psi_0(x) + C_2\psi_0^*(x),$$

where C_1 and C_2 are two known complex constants.

1. Write down the condition for the normalization of $\psi(x)$ in terms of the known complex integral,

$$D = \int_{-\infty}^{\infty} \psi_0(x)\psi_0(x)\,dx.$$

2. Obtain an expression for the probability current $J(x)$ for the state $\psi(x)$. Use a polar representation of complex numbers, i.e. $\psi_0(x) = f(x)e^{i\theta(x)}$.

3. Calculate the expectation value of the momentum $\langle p\rangle$, and show that

$$\langle\psi|\hat{p}\psi\rangle = m\int_{-\infty}^{\infty} J(x)\,dx.$$

   Show that both probability current and momentum vanish if $|C_1| = |C_2|$.

Solution:

1. The normalization condition is

$$\begin{aligned}\langle\psi|\psi\rangle &= \langle C_1\psi_0(x) + C_2\psi_0^*(x)|C_1\psi_0(x) + C_2\psi_0^*(x)\rangle \\ &= |C_1|^2\langle\psi_0|\psi_0\rangle + C_1C_2^*\langle\psi_0^*|\psi_0\rangle + C_1^*C_2\underbrace{\langle\psi_0|\psi_0^*\rangle}_{\langle\psi_0^*|\psi_0\rangle^*} + |C_2|^2\langle\psi_0^*|\psi_0^*\rangle \\ &= |C_1|^2 + C_1C_2^*D + C_1^*C_2D^* + |C_2|^2,\end{aligned}$$

   thus

$$\boxed{\langle\psi|\psi\rangle = |C_1|^2 + |C_2|^2 + 2\mathrm{Re}\left[C_1C_2^*D\right] = 1}.$$

2. Using the polar representation we have

$$\psi_0(x) = f(x)e^{i\theta(x)} \quad\rightarrow\quad \psi(x) = f(x)\left(C_1e^{i\theta(x)} + C_2e^{-i\theta(x)}\right)$$

   where $f(x)$ and $\theta(x)$ are a real valued functions. Therefore the differentiation of ψ reads

$$\frac{\partial\psi}{\partial x} = f'(x)\left(C_1e^{i\theta(x)} + C_2e^{-i\theta(x)}\right) + i\theta'(x)f(x)\left(C_1e^{i\theta(x)} - C_2e^{-i\theta(x)}\right).$$

   The probability current is

$$J(x) = \frac{\hbar}{m}\mathrm{Im}\left[\psi^*\frac{\partial\psi}{\partial x}\right],$$

   where

$$\begin{aligned}\psi^*\frac{\partial\psi}{\partial x} &= f(x)\left(C_1^*e^{-i\theta(x)} + C_2^*e^{i\theta(x)}\right)\left(f'(x)\left(C_1e^{i\theta(x)} + C_2e^{-i\theta(x)}\right) + i\theta'(x)f(x)\left(C_1e^{i\theta(x)} - C_2e^{-i\theta(x)}\right)\right) \\ &= ff'\left(C_1^*e^{-i\theta} + C_2^*e^{i\theta}\right)\left(C_1e^{i\theta} + C_2e^{-i\theta}\right) + i\theta'f^2\left(C_1^*e^{-i\theta} + C_2^*e^{i\theta}\right)\left(C_1e^{i\theta} - C_2e^{-i\theta}\right) \\ &= ff'\left(|C_1|^2 + C_1C_2^*e^{2i\theta} + C_1^*C_2e^{-2i\theta} + |C_2|^2\right) + i\theta'f^2\left(|C_1|^2 + C_1C_2^*e^{2i\theta} - C_1^*C_2e^{-2i\theta} - |C_2|^2\right) \\ &= ff'\underbrace{\left(|C_1|^2 + 2\mathrm{Re}\left[C_1C_2^*e^{2i\theta}\right] + |C_2|^2\right)}_{\text{rael}} + i\theta'f^2\left(|C_1|^2 - |C_2|^2 + 2i\mathrm{Im}\left[C_1C_2^*e^{2i\theta}\right]\right),\end{aligned}$$

   taking only the imaginary part of $\psi^*\frac{\partial\psi}{\partial x}$, we find

$$\boxed{J(x) = \frac{\hbar}{m}\theta'(x)f^2(x)\left(|C_1|^2 - |C_2|^2\right)}.$$

3. The momentum expectation value is

$$\langle\psi|\hat{p}\psi\rangle = -i\hbar\int_{-\infty}^{\infty}\psi^*\psi' dx.$$

But before we go into calculating this in terms of $f(x)$ and $\theta(x)$, let us recall that we already calculated $\psi^*\psi'$ in terms of the real functions $f(x)$ and $\theta(x)$, and that $\langle p\rangle$ must be *real*! Therefore, the only term that can be in the answer is the imaginary part of $\psi^*\psi'$,

$$\boxed{\langle\psi|\hat{p}\psi\rangle = \hbar\left(|C_1|^2 - |C_2|^2\right)\int_{-\infty}^{\infty}\theta' f^2 dx},$$

which is just $J(x)$, up to factor of m,

$$\boxed{\langle\psi|\hat{p}\psi\rangle = m\int_{-\infty}^{\infty}J(x)\,dx}.$$

It is straightforward that both $J(x)$ and $\langle p\rangle$ vanish when $|C_1| = |C_2|$, i.e. when the wave function $\psi(x)$ is real.

Question 3:

Consider a free particle described by the wave function

$$\Psi(x,t) = \left(Ae^{i\frac{p}{\hbar}x} + Be^{-i\frac{p}{\hbar}x}\right)e^{-i\frac{E}{\hbar}t},$$

where $E = p^2/2m$. Find the probability current $J(x,t)$ associated with this wave function. Interpret the different terms and show that if $|A| = |B|$ the probability current vanishes.

Solution:

In order to calculate $J(x,t) = \frac{\hbar}{m}\text{Im}\left[\Psi^*\frac{\partial\Psi}{\partial x}\right]$, we calculate $\Psi^*\frac{\partial\Psi}{\partial x}$,

$$\frac{\partial\Psi}{\partial x} = i\frac{p}{\hbar}\left[Ae^{i\frac{p}{\hbar}x} - Be^{-i\frac{p}{\hbar}x}\right]e^{-i\frac{E}{\hbar}t},$$

thus

$$\begin{aligned}\Psi^*\frac{\partial\Psi}{\partial x} &= i\frac{p}{\hbar}\left(A^*e^{-i\frac{p}{\hbar}x} + B^*e^{i\frac{p}{\hbar}x}\right)\left(Ae^{i\frac{p}{\hbar}x} - Be^{-i\frac{p}{\hbar}x}\right)\\ &= i\frac{p}{\hbar}\left(|A|^2 + 2i\text{Im}\left[AB^*e^{i\frac{2p}{\hbar}x}\right] - |B|^2\right).\end{aligned}$$

Therefore

$$\boxed{J(x,t) = \frac{p}{m}\left(|A|^2 - |B|^2\right)}.$$

The interpretation is that the wave function is made out of two states of free a free particle: one propagates to the positive x direction and the other to the negative x direction. Therefore, when both amplitudes are the same we get a cancellation of the probability current.

Question 4:

Consider a particle that is free to move on a ring of circumference L, such that $\Psi(x,t) = \Psi(x+L,t)$.

1. Find the normalized stationary states of the system and explicitly show that they form an orthonormal basis.
2. In the sum $\psi(x) = \sum c_n\psi_n(x)$, what condition must apply on the coefficients c_n so that $\psi(x)$ would be a "legal" quantum state?

3. Show that any linear combination of the stationary states is also a solution of the (time-dependent) Schrödinger equation if $\omega_n = \frac{\hbar k_n^2}{2m}$.
4. Calculate the dispersion relation $\omega_n(k_n)$ and show that $\omega_n = \omega_{-n}$.
5. Given some initial condition $\Psi(x,0) = f(x)$, where f is some given normalized function, find a way to calculate the coefficients c_n.

Solution:

1. Writing the stationary wave function $\psi(x)$ as a combination of plane waves

$$\psi(x) \propto e^{\pm ikx},$$

we find that

$$\psi(x) = \psi(x+L) \quad \rightarrow \quad e^{ikL} = 1 \quad \rightarrow \quad k \equiv k_n = \frac{2\pi n}{L}.$$

meaning that the stationary states of the system are a discrete set of plane waves

$$\boxed{\psi_n(x) = N_n e^{ik_n x}, \qquad \text{with } k_n = \frac{2\pi n}{L}},$$

for which, the normalization factor is

$$\boxed{\langle \psi_n, \psi_n \rangle = \int_0^L |N_n|^2 \, dx = N_n L = 1 \quad \text{if} \quad |N_n| = \frac{1}{\sqrt{L}}}.$$

2. Considering any linear combination of states,

$$\psi(x) = \sum_n c_n \psi_n(x),$$

the normalization condition states that

$$\langle \psi, \psi \rangle = \sum_{n,m} c_n^* c_m \langle \psi_n, \psi_m \rangle = \sum_{n,m} c_n^* c_m \delta_{nm} = \boxed{\sum_n |c_n|^2 = 1}.$$

3. Since we are dealing with a free particle, the Schrödinger equation reads

$$i\hbar \frac{\partial \Psi}{\partial t} = -\frac{\hbar^2}{2m} \frac{\partial^2 \Psi}{\partial x^2}.$$

Plugging in the time dependent wave function

$$\Psi(x,t) = \sum_n c_n \psi_n(x) e^{-i\omega_n t},$$

we find

$$\hbar \omega \Psi = \frac{\hbar^2 k_n^2}{2m} \Psi \quad \rightarrow \quad \boxed{\omega = \omega_n = \frac{\hbar k_n^2}{2m}}.$$

4. Plugging in the expression for k_n we find

$$\boxed{\omega_n = \frac{2\pi^2 \hbar n^2}{L^2 m} = \omega_{-n}}.$$

5. Considering the initial condition we have

$$f(x) = \sum_m c_m \psi_m(x).$$

Using the orthonormality of ψ_n we can write

$$\boxed{\langle f | \psi_n \rangle = \sum_m c_m \langle \psi_m | \psi_n \rangle = c_n}.$$