

# Homework 5 - Variables Separation, Infinite Square Well

## Question 1:

Following the previous question, consider a particle on a ring that is initially in a state

$$\psi(x) = A \left( \psi_{-2}(x) + \sqrt{2}\psi_{-1}(x) + \psi_0(x) + \sqrt{3}\psi_1(x) + \sqrt{2}\psi_2(x) \right),$$

1. Find  $|A|$ .
2. Find  $\Psi(x, t)$ .
3. We define the  $\hat{Z}$  operator as

$$\hat{Z}\psi_n(x) = \frac{1 + (-1)^n}{2} \psi_n(x).$$

What is the meaning of this operator? Find the probability for having  $\langle Z \rangle = 1$ .

4. Find the expectation value for the particle's energy  $\langle \hat{H} \rangle$ , show that it is time-independent.
5. Find the probability of measuring  $p = 4\pi\hbar/L$ .
6. Find the probability of measuring  $p = 2\pi\hbar/L$ .

At some time  $t_0$  we measure  $\hat{Z}$  of the wave function and get  $z = 1$ .

7. Write down the wave function  $\Psi(x, t > t_0)$ . Make sure to normalize it properly.
8. Repeat articles (4) (5) and (6) for the system at times after  $t_0$ .

## Solution:

1. Using the normalization condition

$$\langle \psi | \psi \rangle = |A|^2 (1 + 2 + 1 + 3 + 2) = 1 \quad \rightarrow \quad \boxed{|A| = \frac{1}{3}}.$$

2. The time dependent wave function is then

$$\boxed{\Psi(x, t) = \frac{1}{3} \left[ \psi_0(x) + \left( \sqrt{2}\psi_{-1}(x) + \sqrt{3}\psi_1(x) \right) e^{-i\frac{E_1}{\hbar}t} + \left( \sqrt{2}\psi_2(x) + \psi_{-2}(x) \right) e^{-i\frac{E_2}{\hbar}t} \right]},$$

where we defined  $E_n = \frac{2\pi^2\hbar^2}{mL^2}n^2$  as the energy of the  $n$ 'th state.

3. It is easy to see that the eigenvalue of the  $\hat{Z}$  operator is 1 if  $n$  is even and 0 if it is odd, so we may call it the state "evenness" operator. The probability for measuring  $\langle Z \rangle = 1$  is

$$\boxed{\langle \Psi | \hat{Z} \Psi \rangle = \frac{1}{9} (\langle \psi_0 | \psi_0 \rangle + 2 \langle \psi_2 | \psi_2 \rangle + \langle \psi_{-2} | \psi_{-2} \rangle) = \frac{4}{9}}.$$

4. Recalling that the Hamiltonian of a free particle is

$$\hat{H} = \frac{\hat{p}^2}{2m} = -\frac{\hbar^2}{2m} \frac{\partial^2}{\partial x^2},$$

and using the Schrödinger equation

$$i\hbar \frac{\partial \Psi}{\partial t} = -\frac{\hbar^2}{2m} \frac{\partial^2 \Psi}{\partial x^2} \rightarrow \hat{H}\Psi = E_n \Psi.$$

Therefore,

$$\langle \Psi | \hat{H} \Psi \rangle = E_n \langle \Psi | \Psi \rangle = \frac{1}{9} (E_0 + 5E_1 + 3E_2) = \frac{17}{9} \frac{2\pi^2 \hbar^2}{mL^2}.$$

5. Looking at the eigenvalue of the momentum operator at a certain state

$$\hat{p}\Psi_n = \hbar k_n \Psi_n = \frac{2\pi \hbar n}{L} \Psi_n,$$

we can easily the relation between  $\langle p \rangle$  and  $n$ , then

$$P\left(\langle p \rangle = \frac{4\pi \hbar}{L}\right) = P(n = 2) = \frac{2}{9}.$$

6. In the same way we have

$$P\left(\langle p \rangle = \frac{2\pi \hbar}{L}\right) = P(n = 1) = \frac{3}{9}.$$

7. Once a measurement was made, the wave function collapses to the state that yields  $\langle Z \rangle = 1$ , thus

$$\Psi(x, t > t_0) = N_{t>t_0} \left[ \psi_0(x) + \left( \sqrt{2}\psi_2(x) + \psi_{-2}(x) \right) e^{-i\frac{E_2}{\hbar}t} \right],$$

such that

$$\langle \Psi(x, t > t_0) | \Psi(x, t > t_0) \rangle = |N_{t>t_0}|^2 (1 + 3) = 1,$$

thus

$$\Psi(x, t > t_0) = \frac{1}{2} \left[ \psi_0(x) + \left( \sqrt{2}\psi_2(x) + \psi_{-2}(x) \right) e^{-i\frac{E_2}{\hbar}t} \right].$$

8. Repeating the (4-6) we find

$$\langle \hat{H} \rangle_{t>t_0} = \frac{1}{4} (E_0 + 3E_2) = 3 \frac{2\pi^2 \hbar^2}{mL^2},$$

$$P\left(\langle p \rangle = \frac{4\pi \hbar}{L}\right) = P(n = 2) = \frac{1}{2},$$

$$P\left(\langle p \rangle = \frac{2\pi \hbar}{L}\right) = P(n = 1) = 0.$$

## Question 2:

Consider a particle of mass  $m$  confined in a symmetric one-dimensional potential well of width  $a$

$$V(x) = \begin{cases} 0, & |x| < \frac{a}{2}, \\ \infty, & \text{else.} \end{cases}$$

1. Find the eigenstates of the Hamiltonian, i.e. stationary states, and the corresponding energies.

- How does the width of the well  $a$  affects the difference between energy levels?
- Do the eigenstates you found in (1) have a defined parity? How does it depend on  $n$ ?
- Find the expectation value for the position of the particle when the system is prepared in a specific eigenstate.

**Solution:**

- We have already solved the problem of an infinite square well at  $0 < x < a$ , and got the stationary states

$$\psi_n(x') = \begin{cases} \sqrt{\frac{2}{a}} \sin\left(\frac{n\pi}{a}x'\right), & 0 \leq x' \leq a, \\ 0 & \text{else,} \end{cases} \quad \text{with} \quad E_n = \frac{\hbar^2 \pi^2 n^2}{2ma^2}.$$

Then, all we have to do is to set  $x' = x + a/2$ , so we get

$$\psi_n(x) = \begin{cases} \sqrt{\frac{2}{a}} \sin\left(\frac{n\pi}{a}x + \frac{n\pi}{2}\right), & -\frac{a}{2} \leq x \leq \frac{a}{2}, \\ 0 & \text{else,} \end{cases} \quad \text{with} \quad E_n = \frac{\hbar^2 \pi^2 n^2}{2ma^2}.$$

- The difference between two sequential energy levels is

$$\Delta E = E_{n+1} - E_n = \frac{\hbar^2 \pi^2}{2ma^2} (2n+1) \propto a^{-2}.$$

- As we mentioned before, for symmetric potential, even and odd states correspond to odd and even state functions. It can be easily seen in this example, since without the  $n\pi/2$  term in the sine, it is an odd function around the origin. Then, by adding a phase of odd multiplication of  $\pi/2$  we get a cosine which is even around the origin, whereas even multiplication correspond to phases of multiplication of  $\pi$  which does not change the oddness of the sine function.

$$\begin{aligned} \text{odd } n &\rightarrow \psi(-x) = \psi(x), \\ \text{even } n &\rightarrow \psi(-x) = -\psi(x). \end{aligned}$$

- If the system was prepared in a certain state  $\psi_n$  then

$$\langle \psi_n | \hat{x} \psi_n \rangle = \int_{-\infty}^{\infty} \psi_n^* x \psi_n dx = \frac{2}{a} \int_{-a/2}^{a/2} x \sin^2\left(\frac{n\pi}{a}x + \frac{n\pi}{2}\right) dx,$$

but  $\sin^2(x + n\pi/2)$  is even (it is either  $\cos^2 x$  or  $\sin^2 x$ , depends on  $n$  being odd or even number), hence the integral vanishes and we get

$$\langle \psi_n | \hat{x} \psi_n \rangle = 0,$$

which is what we expect from a symmetric potential around the origin.

### Question 3:

Consider an infinite square well of width  $2L$ , with a particle of mass  $m$  moving in it ( $|x| < L$ ). It is given that the particle is in the lowest-energy state at  $t < 0$ . Assume now that at  $t = 0$  the walls of the well move instantaneously such that the well doubles ( $|x| < 2L$ ). This transition of the system does not affect the state of the particle, which remains the same at the moment of the transition.

- Write down the wave function  $\Psi(x, 0)$  (before the transition) and find the corresponding energy of the particle.
- Write down the wave function of the particle at times  $t > 0$ . Calculate the probability  $P_n$  of finding the particle in an arbitrary eigenstate of the modified system. What is the probability of finding the particle in an odd eigenfunction (with respect to  $x$ )?

3. Calculate the expectation value of the energy at any time  $t > 0$ . You can make use of the series

$$\sum_{\nu=0}^{\infty} \frac{(2\nu+1)^2}{[(2\nu+1)^2-4]^2} = \frac{\pi^2}{16}.$$

4. If we assume that the walls move outwards with a finite speed  $u$ , our assumptions should still hold provided that this velocity is much larger than the characteristic velocity of the system at times  $t < 0$ , i.e.  $u \ll v_0$ . What is  $v_0$ ?

**Solution:**

1. The solution for infinite well of width  $2L$  at the lowest energy level ( $n = 1$ ) is

$$\psi(x, 0) = \psi_1(x) = \sqrt{\frac{1}{L}} \cos\left(\frac{\pi x}{2L}\right), \quad \text{with} \quad E(t < 0) = E_1 = \frac{\hbar^2 \pi^2}{8mL^2}.$$

2. After the transition of the system, the stationary states become

$$\psi_n(x) = \sqrt{\frac{1}{2L}} \sin\left(\frac{n\pi}{4L}x + \frac{n\pi}{2}\right), \quad \text{with} \quad E_n = \frac{\hbar^2 \pi^2 n^2}{32mL^2} = \frac{1}{4}E(t < 0),$$

and the general solution is

$$\Psi(x, t < 0) = \sum_n c_n \psi_n e^{-iE_n t/\hbar}.$$

In order to find the  $c_n$ s we must use the initial wave function, so that

$$c_n = \int_{-\infty}^{\infty} \psi_1^*(x) \psi_n(x) dx = \frac{1}{\sqrt{2L}} \int_{-L}^L \cos\left(\frac{\pi x}{2L}\right) \sin\left(\frac{n\pi}{4L}x + \frac{n\pi}{2}\right) dx,$$

at this point it is clear that for even  $n$ s the integral vanishes (symmetric integral of an odd function), therefore we can already limit ourselves to odd  $n$ s ( $n \rightarrow 2n - 1$ ). Thus

$$\begin{aligned} c_{2n-1} &= \frac{(-1)^{n+1}}{\sqrt{2L}} \int_{-L}^L \cos\left(\frac{\pi x}{2L}\right) \cos\left(\frac{(2n-1)\pi}{4L}x\right) dx \\ &= \frac{(-1)^{n+1}}{2\sqrt{2L}} \int_{-L}^L \left[ \cos\left(\frac{(2n-3)\pi}{4L}x\right) + \cos\left(\frac{(2n+1)\pi}{4L}x\right) \right] dx \\ &= \frac{2(-1)^{n+1}}{\sqrt{2\pi}} \left[ \frac{1}{2n-3} \sin\left(\frac{(2n-3)\pi}{4L}x\right) + \frac{1}{2n+1} \sin\left(\frac{(2n+1)\pi}{4L}x\right) \right] \Big|_{-L}^L \\ &= \frac{2\sqrt{2}(-1)^{n+1}}{\pi} \left[ \frac{1}{2n-3} \sin\left(\frac{(2n-3)\pi}{4}\right) + \frac{1}{2n+1} \sin\left(\frac{(2n+1)\pi}{4}\right) \right] \\ &= \frac{2\sqrt{2}(-1)^{n+1}}{\pi} \left[ -\frac{1}{2n-3} + \frac{1}{2n+1} \right] \sin\left(\frac{(2n+1)\pi}{4}\right) \\ &= \frac{8\sqrt{2}(-1)^n}{\pi(2n-3)(2n+1)} \sin\left(\frac{(2n+1)\pi}{4}\right). \end{aligned}$$

The probability of finding the particle at some state is

$$P_n = |c_n|^2 = \frac{128}{\pi^2 (n-2)^2 (n+2)^2} \sin^2\left(\frac{(n+2)\pi}{4}\right) = \frac{64}{\pi^2 (n^2-4)^2}, \quad \text{where} \quad n = 1, 3, 5, \dots$$

3. The expectation value of the energy is

$$\begin{aligned} \langle \hat{H} \rangle &= \sum_n |c_n|^2 E_n \\ &= \frac{2\hbar^2}{mL^2} \sum_{\text{odd } n} \frac{n^2}{(n^2-4)^2} \\ &= \frac{2\hbar^2}{mL^2} \sum_{\nu} \frac{(2\nu+1)^2}{[(2\nu+1)^2-4]^2}, \end{aligned}$$

thus

$$\boxed{\langle \hat{H} \rangle = \frac{\pi^2 \hbar^2}{2m(2L)^2}}$$

4. Before the transition, the characteristic velocity of the system was

$$\boxed{v_0 = \sqrt{\frac{2E(t < 0)}{m}} = \frac{\hbar\pi}{2mL}}$$

thus, the walls must move much faster than  $v_0$ .

### Question 4:

Show that if the function  $\Psi(x, t)$  is a solution of the Schrödinger equation then so is the function  $\Psi'(x, t) = \Psi(x, t)e^{i\varphi}$  where  $\varphi$  is some arbitrary phase. Do we have to renormalize  $\Psi'(x, t)$  after this transition? How does it affect the probability density  $\rho(x, t)$  and the probability current density  $J(x, t)$ ?

**Solution:**

The time-dependent Schrödinger equation reads

$$i\hbar \frac{\partial}{\partial t} \Psi = -\frac{\hbar^2}{2m} \frac{\partial^2}{\partial x^2} \Psi + V\Psi,$$

plugging  $\Psi \rightarrow \Psi e^{i\varphi}$ , changes nothing since

$$i\hbar \frac{\partial}{\partial t} (\Psi e^{i\varphi}) = e^{i\varphi} i\hbar \frac{\partial}{\partial t} \Psi \quad \text{and} \quad -\frac{\hbar^2}{2m} \frac{\partial^2}{\partial x^2} (\Psi e^{i\varphi}) + V(\Psi e^{i\varphi}) = -e^{i\varphi} \frac{\hbar^2}{2m} \frac{\partial^2}{\partial x^2} \Psi + e^{i\varphi} V\Psi,$$

hence the phase cancels out and the equation remains the same.

Since the norm of the phase  $e^{i\varphi}$  is 1, there is no need to renormalize the wave function,

$$\langle \Psi | \Psi \rangle \rightarrow \langle \Psi e^{i\varphi} | \Psi e^{i\varphi} \rangle = e^{-i\varphi} e^{i\varphi} \langle \Psi | \Psi \rangle = \langle \Psi | \Psi \rangle = 1.$$

In the same manner

$$\rho(x, t) = |\Psi|^2 \rightarrow |\Psi e^{i\varphi}|^2 = |\Psi|^2 = \rho,$$

and

$$J(x, t) = \frac{\hbar}{m} \text{Im} \left[ \Psi^* \frac{\partial}{\partial x} \Psi \right] \rightarrow \frac{\hbar}{m} \text{Im} \left[ (\Psi e^{i\varphi})^* \frac{\partial}{\partial x} (\Psi e^{i\varphi}) \right] = \frac{\hbar}{m} \text{Im} \left[ \Psi^* \frac{\partial}{\partial x} \Psi \right] = J(x, t).$$