Homework 7 - Infinite Square Well

Question 1:

Consider a particle of mass m confined in a symmetric one-dimensional potential well. The initial state of the particle is given by the wave function

$$\Psi(x,0)=\begin{cases} b+\frac{2b}{a}x, & -\frac{a}{2}\le x\le 0,\\ b-\frac{2b}{a}x, & 0\le x\le \frac{a}{2}.\end{cases}$$

1. Find b.
2. What are the probabilities of measuring any of the five lowest energies of the system?
3. Calculate the mean value of the particles energy $\langle E\rangle$.
4. Find the state of the system at time t.

Solution:

1. Using the normalization condition $\langle\Psi|\Psi\rangle=1$, we have

$$\begin{aligned}\langle\Psi(x,0)|\Psi(x,0)\rangle &= |b|^2\left[\int_{-a/2}^{0}\left(1+\frac{2}{a}x\right)^2dx+\int_0^{a/2}\left(1-\frac{2}{a}x\right)^2dx\right]\\ &= |b|^2\left[\left(x+\frac{2}{a}x^2+\frac{4}{3a^2}x^3\right)\Big|_{-a/2}^{0}+\left(x-\frac{2}{a}x^2+\frac{4}{3a^2}x^3\right)\Big|_0^{a/2}\right]\\ &= |b|^2\left[-\left(-\frac{a}{2}+\frac{a}{2}-\frac{a}{6}\right)+\left(\frac{a}{2}-\frac{a}{2}+\frac{a}{6}\right)\right]\\ &= \frac{a}{3}|b|^2,\end{aligned}$$

hence

$$\boxed{|b|=\sqrt{\frac{3}{a}}}.$$

2. The eigenstates for an infinite well potential at $0\le x\le a$ are

$$\psi_n(x)=\sqrt{\frac{2}{a}}\sin\left(\frac{\pi n x}{a}\right),\qquad \text{with the associated energies}\qquad E_n=\frac{\hbar^2\pi^2n^2}{2ma^2},$$

and the general solution is of the form

$$\psi(x)=\sum_n c_n\psi_n(x),$$

and the probability to measure the E_n is $|c_n|^2$. Therefore, let us use this coordinate and write

$$\Psi(x,0)=\sqrt{\frac{3}{a}}\begin{cases}\frac{2}{a}x, & 0\le x\le\frac{a}{2},\\ 2\left(1-\frac{x}{a}\right), & \frac{a}{2}\le x\le a,\end{cases}$$

so that

$$c_n=\langle\psi_n|\Psi(x,0)\rangle=\sqrt{\frac{2}{a}}\sqrt{\frac{3}{a}}\left[\int_0^{a/2}\frac{2}{a}x\sin\left(\frac{\pi n x}{a}\right)dx+\int_{a/2}^{a}2\left(1-\frac{x}{a}\right)\sin\left(\frac{\pi n x}{a}\right)dx\right].$$

The integrals are

$$\begin{aligned}
\int_0^{a/2} \frac{2}{a} x \sin\left(\frac{\pi n x}{a}\right) dx &= -\frac{2}{\pi n}\int_0^{a/2} x \frac{d}{dx}\left[\cos\left(\frac{\pi n x}{a}\right)\right] dx \\
&= -\frac{2}{\pi n}\left[x\cos\left(\frac{\pi n x}{a}\right)\right]_0^{a/2} + \frac{2}{\pi n}\int_0^{a/2}\cos\left(\frac{\pi n x}{a}\right) dx \\
&= -\frac{a}{\pi n}\cos\left(\frac{\pi n}{2}\right) + 2\frac{a}{\pi^2 n^2}\sin\left(\frac{\pi n}{2}\right), \\
\int_{a/2}^{a} 2\left(1-\frac{x}{a}\right)\sin\left(\frac{\pi n x}{a}\right) dx &= 2\int_{a/2}^{a}\sin\left(\frac{\pi n x}{a}\right) dx - \int_{a/2}^{a}\frac{2}{a} x \sin\left(\frac{\pi n x}{a}\right) dx \\
&= -2\frac{a}{n\pi}\left(\cos(\pi n) - \cos\left(\frac{\pi n}{2}\right)\right) - \left(\frac{2}{\pi n}\left[-x\cos\left(\frac{\pi n x}{a}\right)\right]_{a/2}^{a} + \frac{2}{\pi n}\int_{a/2}^{a}\cos\left(\frac{\pi n x}{a}\right) dx\right) \\
&= -2\frac{a}{n\pi}\left(\cos(\pi n) - \cos\left(\frac{\pi n}{2}\right)\right) - \left(\frac{2}{\pi n}\left[-a\cos(\pi n) + \frac{a}{2}\cos\left(\frac{\pi n}{2}\right)\right] - \frac{2a}{\pi^2 n^2}\sin\left(\frac{\pi n}{2}\right)\right) \\
&= \frac{a}{n\pi}\cos\left(\frac{\pi n}{2}\right) + \frac{2a}{\pi^2 n^2}\sin\left(\frac{\pi n}{2}\right),
\end{aligned}$$

therefore

$$c_n = \frac{4\sqrt{6}}{\pi^2 n^2}\sin\left(\frac{\pi n}{2}\right) \quad \rightarrow \quad \boxed{P\left(\langle E\rangle = E_n\right) = \frac{96}{\left(n^2\pi^2\right)^2}\sin^2\left(\frac{n\pi}{2}\right)}.$$

In particular

$$\boxed{P_{2n} = 0 \quad \text{and} \quad P_{2n+1} = \frac{96}{(2n+1)^4\pi^4}},$$

$$\begin{aligned}
P_0 &= P_2 = P_4 = 0, \\
P_1 &= \frac{96}{\pi^4} \approx 0.9855, \\
P_3 &= \frac{96}{81\pi^4} \approx 0.0122, \\
P_5 &= \frac{96}{625\pi^4} \approx 0.0016.
\end{aligned}$$

3. The mean value is simply

$$\begin{aligned}
\langle E\rangle &= \sum_n P_n E_n \\
&= \sum_n \left|\langle\psi_n|\Psi\rangle\right|^2 E_n \\
&= \sum_k \frac{96}{(2k+1)^4\pi^4}\frac{\hbar^2\pi^2(2k+1)^2}{2ma^2} \\
&= \frac{48\hbar^2}{m\pi^2 a^2}\sum_k \frac{1}{(2k+1)^2},
\end{aligned}$$

hence

$$\boxed{\langle E\rangle = \frac{6\hbar^2}{ma^2}}.$$

4. At time t we have

$$\boxed{\Psi(x,t) = \sum_n c_n\psi_n(x)\, e^{-iE_n t/\hbar} = \sum_k \frac{8}{\pi^2(2k+1)^2}\sqrt{\frac{3}{a}}\sin\left(\frac{\pi(2k+1)x}{a}\right) e^{-iE_{2k+1}t/\hbar}}.$$

Question 2:

Consider a particle of mass m confined in a symmetric one-dimensional potential well as in the previous question. The initial state of the particle is given by the wave function

$$\Psi(x,0) = A(\varphi_1 - \varphi_2),$$

where φ_n are the eigenstates of the symmetric infinite well.

1. Find A.
2. Find the state of the system at time t.
3. Calculate the probability current density $J(x,t)$.
4. Calculate the mean value of the particles position $\langle x\rangle$ at time t.
5. Calculate the mean value of the particles momentum $\langle p\rangle$ at time t.

Solution:

1. Using normalization condition we find

$$\langle\Psi(x,0)|\Psi(x,0)\rangle = |A|^2\left(\langle\varphi_1|\varphi_1\rangle - \cancel{\langle\varphi_1|\varphi_2\rangle} - \cancel{\langle\varphi_2|\varphi_1\rangle} + \langle\varphi_2|\varphi_2\rangle\right) = 2|A|^2 = 1,$$

   hence

$$\boxed{|A| = \frac{1}{\sqrt{2}}}.$$

2. As before,

$$\Psi(x,t) = \sum c_n\varphi_n(x)e^{-iE_nt/\hbar}, \qquad \text{where } E_n = \frac{\hbar^2\pi^2n^2}{2ma^2} \qquad \text{and } \varphi_n(x) = \sqrt{\frac{2}{a}}\sin\left(\frac{n\pi}{a}x + \frac{n\pi}{2}\right)$$

   and

$$c_n = \langle\varphi_n|\Psi(x,0)\rangle = \begin{cases} \frac{1}{\sqrt{2}}, & n=1, \\ -\frac{1}{\sqrt{2}}, & n=2. \end{cases}$$

   Therefore

$$\boxed{\Psi(x,t) = \frac{1}{\sqrt{2}}\left(\varphi_1e^{-iE_1t/\hbar} - \varphi_2e^{-iE_2t/\hbar}\right) = \frac{1}{\sqrt{a}}\left(\cos\left(\frac{\pi}{a}x\right)e^{-iE_1t/\hbar} + \sin\left(\frac{2\pi}{a}x\right)e^{-iE_2t/\hbar}\right)}.$$

3. Recalling that

$$J(x,t) = \frac{\hbar}{m}\text{Im}\left[\Psi^*\frac{\partial\Psi}{\partial x}\right],$$

   where

$$\frac{\partial\Psi}{\partial x} = \frac{1}{\sqrt{2}}\left(\varphi_1'e^{-iE_1t/\hbar} - \varphi_2'e^{-iE_2t/\hbar}\right) = \frac{\pi}{a^{3/2}}\left(-\sin\left(\frac{\pi}{a}x\right)e^{-iE_1t/\hbar} + 2\cos\left(\frac{2\pi}{a}x\right)e^{-iE_2t/\hbar}\right),$$

   thus

$$\Psi^*\frac{\partial\Psi}{\partial x} = -\frac{i\pi}{a^2}\left[\sin\left(\frac{\pi}{a}x\right)\sin\left(\frac{2\pi}{a}x\right) + 2\cos\left(\frac{2\pi}{a}x\right)\cos\left(\frac{\pi}{a}x\right)\right]\sin\left(\frac{E_2-E_1}{\hbar}t\right) + \text{real terms}.$$

$$\boxed{J(x,t) = -\frac{\hbar\pi}{2ma^2}\left[3\cos\left(\frac{\pi x}{a}\right) + \cos\left(\frac{3\pi x}{a}\right)\right]\sin\left(\frac{3\hbar\pi^2}{2ma^2}t\right)},$$

   where we used the fact that $E_2 - E_1 = \frac{3\hbar^2\pi^2}{2ma^2}$.

4. The mean value of the position is

$$\begin{aligned}\langle x\rangle &= \langle\Psi|\hat{x}\Psi\rangle \\ &= \frac{1}{a}\int_{-a/2}^{a/2}\left(\cos\left(\frac{\pi}{a}x\right)e^{iE_1t/\hbar}+\sin\left(\frac{2\pi}{a}x\right)e^{iE_2t/\hbar}\right)x\left(\cos\left(\frac{\pi}{a}x\right)e^{-iE_1t/\hbar}+\sin\left(\frac{2\pi}{a}x\right)e^{-iE_2t/\hbar}\right)dx \\ &= \frac{1}{a}\int_{-a/2}^{a/2}x\cos\left(\frac{\pi}{a}x\right)\sin\left(\frac{2\pi}{a}x\right)\left(e^{i(E_2-E_1)t/\hbar}+e^{-i(E_2-E_1)t/\hbar}\right)dx \\ &= \frac{1}{a}\cos\left(\frac{3\hbar\pi^2}{2ma^2}t\right)\int_{-a/2}^{a/2}x\left[\sin\left(\frac{\pi x}{a}\right)+\sin\left(\frac{3\pi x}{a}\right)\right]dx,\end{aligned}$$

where we dropped all the odd terms in the integral. Using the integral

$$\int_{-a/2}^{a/2}x\sin\left(\frac{n\pi x}{a}\right)dx=\frac{a^2}{(n\pi)^2}\left[2\sin\left(\frac{n\pi}{2}\right)-n\pi\cos\left(\frac{n\pi}{2}\right)\right],$$

we find

$$\boxed{\langle x\rangle=\frac{16a}{9\pi^2}\cos\left(\frac{3\hbar\pi^2}{2ma^2}t\right)}.$$

5. We can calculate $\langle p\rangle=\langle\Psi|\hat{p}\Psi\rangle$, but it is easier to invoke Ehrenfest theorem and use the previous result,

$$\boxed{\langle p\rangle=m\frac{d}{dt}\langle x\rangle=-\frac{8\hbar}{3a}\sin\left(\frac{3\hbar\pi^2}{2ma^2}t\right)}.$$

Question 3:

Consider a particle in an infinite square well of width a,

1. Show that the wave function returns to its original form after a quantum revival time T and find what is its value. That is: $\Psi(x,t)=\Psi(x,t+T)$ for any state (not just a stationary state).
2. What is the classical revival time, for a particle of energy E bouncing back and forth between the walls?
3. For what energy are the two revival times equal?

Solution:

1. The general solution for a particle in an infinite square well is

$$\Psi(x,t)=\sum_n c_n\psi_n(x)\,e^{-i\frac{\hbar\pi^2n^2}{2ma^2}t},$$

hence we must require

$$\frac{\hbar\pi^2n^2}{2ma^2}T=2\pi\times\underbrace{\text{integer}}_{n^2}\quad\rightarrow\quad\boxed{T=\frac{4ma^2}{\hbar\pi}}.$$

2. The classical revival time is the time it takes the particle to go the distance $2a$, thus

$$T_c=\frac{2a}{v},$$

where

$$v=\sqrt{\frac{2E}{m}}\quad\rightarrow\quad\boxed{T_c=a\sqrt{\frac{2m}{E}}}.$$

3. The two are the same if the energy of the particle equals

$$T_c=T\quad\rightarrow\quad\boxed{E=\frac{\hbar^2\pi^2}{8ma^2}=\frac{E_1}{4}}.$$

Question 4:

Consider an infinite square well of width L, with a particle of mass m moving in it $(-L/2 < x < L/2)$. At time $t = 0$, the state of the particle is described by the wave function

$$\psi(x) = A\left[3\varphi_1(x) + 4\varphi_2(x)\right].$$

1. Find $|A|$.
2. Find $\Psi(x,t)$.
3. We measure the particle's position at time t. What is the probability of finding the particle at the right half of the well?
4. Find $\langle x\rangle(t)$ and $\langle p\rangle(t)$. Notice that while these are periodic, they are very different from the classical results. Discuss the reasons for this difference.

Solution:

For the ease of calculations let us shift our coordinate to $x \to x + L/2$, where the solution for an infinite square well potential in $(0, L)$ is

$$\varphi_n(x) = \sqrt{\frac{2}{L}}\sin\left(\frac{n\pi}{L}x\right), \quad \text{with } E_n = \frac{\pi^2\hbar^2 n^2}{2mL^2}.$$

1. Using the normalization condition

$$\langle\psi|\psi\rangle = |A|^2(9+16) = 1 \quad \to \quad \boxed{|A| = \frac{1}{5}}.$$

2. Since the only eigenstates in the initial state are $n = 1, 2$ we have

$$\boxed{\Psi(x,t) = \frac{3}{5}\varphi_1(x)e^{-iE_1 t/\hbar} + \frac{4}{5}\varphi_2(x)e^{-iE_2 t/\hbar}}.$$

3. The probability for finding the particle at $x \in (L/2, L)$ is

$$\begin{aligned}
P\left(\frac{L}{2} < x < L\right) &= \int_{L/2}^{L}\Psi^*\Psi dx \\
&= \frac{1}{25}\int_{L/2}^{L}\left[9|\varphi_1|^2 + 12\varphi_2^*\varphi_1 e^{i(E_2-E_1)t/\hbar} + 12\varphi_1^*\varphi_2 e^{-i(E_2-E_1)t/\hbar} + 16|\varphi_2|^2\right]dx \\
&= \frac{1}{25}\left[\frac{9}{2} - 12\frac{4}{3\pi}\cos\omega t + \frac{16}{2}\right],
\end{aligned}$$

$$\boxed{P\left(\frac{L}{2} < x < L\right) = \frac{1}{2} - \frac{16}{25\pi}\cos\omega t, \quad \text{with} \quad \omega \equiv \frac{E_2 - E_1}{\hbar} = \frac{3\pi^2\hbar^2}{2mL^2}},$$

where we used the integrals

$$\frac{2}{L}\int_{L/2}^{L}\sin^2\left(\frac{\pi x}{L}\right)dx = \frac{2}{L}\int_{L/2}^{L}\sin^2\left(\frac{2\pi x}{L}\right)dx = \frac{1}{2} \quad \text{and} \quad \frac{2}{L}\int_{L/2}^{L}\sin\left(\frac{\pi x}{L}\right)\sin\left(\frac{2\pi x}{L}\right)dx = -\frac{4}{3\pi}.$$

4. The expectation value of $\hat{x}$ is

$$\begin{aligned}
\langle x\rangle(t) = \langle\Psi|\hat{x}\Psi\rangle &= \left\langle\frac{3}{5}\varphi_1 e^{-iE_1 t/\hbar} + \frac{4}{5}\varphi_2 e^{-iE_2 t/\hbar}\middle|\hat{x}\left(\frac{3}{5}\varphi_1 e^{-iE_1 t/\hbar} + \frac{4}{5}\varphi_2 e^{-iE_2 t/\hbar}\right)\right\rangle \\
&= \frac{9}{25}\langle\varphi_1|\hat{x}\varphi_1\rangle + \frac{12}{25}e^{-i\omega t}\langle\varphi_1|\hat{x}\varphi_2\rangle + \frac{12}{25}e^{i\omega t}\langle\varphi_2|\hat{x}\varphi_1\rangle + \frac{16}{25}\langle\varphi_2|\hat{x}\varphi_2\rangle
\end{aligned}$$

thus, let us first calculate

$$\begin{aligned} \langle \varphi_1 | \hat{x} \varphi_1 \rangle &= \int_0^L \varphi_1^* x \varphi_1 dx = \frac{L}{2}, \\ \langle \varphi_1 | \hat{x} \varphi_2 \rangle &= \int_0^L \varphi_1^* x \varphi_2 dx = -\frac{16L}{9\pi^2}, \\ \langle \varphi_2 | \hat{x} \varphi_2 \rangle &= \int_0^L \varphi_2^* x \varphi_2 dx = \frac{L}{2}. \end{aligned}$$

Plugging these into the expression for $\langle \hat{x} \rangle$ we get

$$\boxed{\langle x \rangle (t) = \frac{9}{25}\frac{L}{2} - \frac{24}{25}\frac{16L}{9\pi^2}\cos\omega t + \frac{16}{25}\frac{L}{2} = L\left(\frac{1}{2} - \frac{128}{75\pi^2}\cos\omega t\right)}.$$

Using Ehrenfest theorem,

$$\boxed{\langle p \rangle (t) = m\frac{d\langle x \rangle}{dt} = \frac{128}{75\pi^2} mL\omega \sin\omega t = \frac{64\hbar^2}{25L}\sin\omega t}.$$