Homework 10 - Operators and Dirac Notation

Question 1:

Consider some physical system that is represented by a Hilbert space with the set of normalized energy eigenstates $\{|n\rangle\,, \quad n = 0, 1, 2, \dots\}$ such that $\hat{H}|n\rangle = E_n |n\rangle$. We define the projector operator on the state $|n\rangle$ by $\hat{P}_n = |n\rangle\langle n|$.

1. Is $\hat{P}_n$ Hermitian?
2. What is the representative matrix of $\hat{P}_n$ in this basis?
3. What are the eigenvalues of $\hat{P}_n$?
4. Prove $\hat{P}_n\hat{P}_m = \delta_{nm}\hat{P}_n$.
5. Is $\hat{P}_n$ unitary?
6. Prove that $\hat{H} = \sum_n E_n \hat{P}_n$.

Solution:

1. Since
$$|n\rangle^\dagger = \langle n| \qquad \text{and} \qquad \langle n|^\dagger = |n\rangle\,,$$
it is straightforward to see that $\hat{P}_n$ is Hermitian:
$$\boxed{\hat{P}_n^\dagger = (|n\rangle\langle n|)^\dagger = \langle n|^\dagger\,|n\rangle^\dagger = |n\rangle\langle n| = \hat{P}_n}.$$

2. Takin the inner product
$$\left\langle k|\hat{P}_n|m\right\rangle = \langle k|n\rangle\langle n|m\rangle = \delta_{kn}\delta_{nm} \quad \rightarrow \quad \boxed{P_n = \begin{pmatrix} 0 & & & & & 0 \\ & \ddots & & & & \\ & & 0 & & & \\ & & & 1 & & \\ & & & & 0 & \\ 0 & & & & & \ddots \end{pmatrix}}.$$

3. When acting with $\hat{P}_n$ on some state $|\mathcal{S}\rangle$, we get
$$\hat{P}_n|\mathcal{S}\rangle = \langle n|\mathcal{S}\rangle\,|n\rangle\,,$$
thus the eigenvalue is $\langle n|\mathcal{S}\rangle$. In the case of energy eigenstates $\{|n\rangle\}$ we find
$$\langle n|m\rangle = \delta_{nm},$$
i.e. the eigenvalues are $\boxed{\text{0 and 1}}$.

4. Writing down $\hat{P}_n\hat{P}_m$ explicitly we find
$$\boxed{\hat{P}_n\hat{P}_m = |n\rangle\langle n|m\rangle\langle m| = \delta_{nm}|n\rangle\langle m| = \delta_{nm}|n\rangle\langle n| = \delta_{nm}\hat{P}_n}.$$

5. Unitarity is defined by $UU^\dagger = U^\dagger U = I$. The projection operator is singular,

$$\det \hat{P}_n = 0,$$

thus it is singular and $\boxed{\text{not unitary}}$.

6. We've shown in class that any operator $\hat{Q}$ can be written in terms of spectral decomposition of orthonormal eigenstate

$$\hat{Q} = \sum_i \lambda_i |e_i\rangle \langle e_i| ,$$

which holds for our case,

$$\boxed{\hat{H} = \sum_n E_n \hat{P}_n}.$$

Question 2:

Consider a harmonic oscillator of mass m and frequency ω.

1. Calculate the matrix elements of the momentum operator using the eigenfunctions of the hamiltonian $p_{kn} = \langle k|\hat{p}|n\rangle$ for $k, n = \{0, 1, 2, 3\}$.
2. Calculate the matrix elements of the momentum operator p_{kn} using ladder operators a and $a^\dagger$.
3. Calculate the matrix elements of the squared momentum operator p^2_{kn} using ladder operators a and $a^\dagger$.
4. Calculate the matrix elements of the squared momentum operator p^2_{kn} using matrix multiplication.
5. Calculate σ_x and σ_p for some eigenfunction of the hamiltonian and calculate $\sigma_x\sigma_p$. What is the minimal value of $\sigma_x\sigma_p$ and what is the corresponding eigenfunction?
6. Find the mean kinetic energy and potential energy of the state $|n\rangle$. Write the solution in terms of the energy eigenvalue E_n. Explain this result.

Solution:

1. The eigenfunctions of the hamiltonian are

$$\psi_n(x) = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} \frac{1}{\sqrt{2^n n!}} H_n\left(\sqrt{\frac{m\omega}{\hbar}}x\right) e^{-\frac{m\omega}{2\hbar}x^2},$$

so that

$$\begin{aligned}
\frac{d}{dx}\psi_n(x) &= \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} \frac{1}{\sqrt{2^n n!}} \left[\frac{d}{dx} H_n\left(\sqrt{\frac{m\omega}{\hbar}}x\right) - \frac{m\omega}{\hbar} x H_n\left(\sqrt{\frac{m\omega}{\hbar}}x\right)\right] e^{-\frac{m\omega}{2\hbar}x^2} \\
&= \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} \frac{1}{\sqrt{2^n n!}} \left[\sqrt{\frac{m\omega}{\hbar}} 2nH_{n-1} - \sqrt{\frac{m\omega}{\hbar}}\left(\frac{1}{2}H_{n+1} + nH_{n-1}\right)\right] e^{-\frac{m\omega}{2\hbar}x^2} \\
&= \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} \frac{1}{\sqrt{2^n n!}} \sqrt{\frac{m\omega}{\hbar}} \left(nH_{n-1} - \frac{1}{2}H_{n+1}\right) e^{-\frac{m\omega}{2\hbar}x^2} \\
&= \sqrt{\frac{m\omega}{2\hbar}} \left(\sqrt{n}\psi_{n-1} - \sqrt{(n+1)}\psi_{n+1}\right)
\end{aligned}$$

where, in the second line, we used the identities

$$\begin{aligned}
\frac{d}{dz} H_n(z) &= 2nH_{n-1}(z), \\
H_n(z) &= 2zH_{n-1}(z) - 2(n-1)H_{n-2}(z).
\end{aligned}$$

Therefore

$$\begin{aligned} p_{kn} &= \langle \psi_k | \hat{p} | \psi_n \rangle \\ &= -i\hbar\sqrt{\frac{m\omega}{2\hbar}} \left\langle \psi_k \right| \left(\sqrt{n}\psi_{n-1} - \sqrt{(n+1)}\psi_{n+1} \right) \Big\rangle , \end{aligned}$$

hence

$$\boxed{p_{kn} = i\sqrt{\frac{m\hbar\omega}{2}} \left(\sqrt{(n+1)}\delta_{k,n+1} - \sqrt{n}\delta_{k,n-1} \right)}.$$

2. Using the ladder operators, we have

$$\hat{p} = i\sqrt{\frac{m\hbar\omega}{2}} \left(\hat{a}^\dagger - \hat{a} \right),$$

thus

$$\begin{aligned} p_{kn} &= \langle k | \hat{p} | n \rangle \\ &= i\sqrt{\frac{m\hbar\omega}{2}} \left(\langle k | \hat{a}^\dagger | n \rangle - \langle k | \hat{a} | n \rangle \right) \\ &= i\sqrt{\frac{m\hbar\omega}{2}} \left(\sqrt{n+1} \langle k | n+1 \rangle - \sqrt{n} \langle k | n-1 \rangle \right), \end{aligned}$$

hence

$$\boxed{p_{kn} = i\sqrt{\frac{m\hbar\omega}{2}} \left(\sqrt{(n+1)}\delta_{k,n+1} - \sqrt{n}\delta_{k,n-1} \right)}.$$

3. In the same fashion of (2) we have

$$\hat{p}^2 = -\frac{m\hbar\omega}{2} \left(\hat{a}^\dagger\hat{a}^\dagger - \hat{a}^\dagger\hat{a} - \hat{a}\hat{a}^\dagger + \hat{a}\hat{a} \right) = \frac{m\hbar\omega}{2} \left(2\hat{a}^\dagger\hat{a} + 1 - \hat{a}^\dagger\hat{a}^\dagger - \hat{a}\hat{a} \right),$$

where in the last equality we used the commutator $\left[\hat{a}, \hat{a}^\dagger\right] = 1$. Therefore

$$\begin{aligned} p^2_{kn} &= \left\langle k | \hat{p}^2 | n \right\rangle \\ &= \frac{m\hbar\omega}{2} \left(2 \left\langle k | \hat{a}^\dagger\hat{a} | n \right\rangle + \langle k | n \rangle - \left\langle k | \hat{a}^\dagger\hat{a}^\dagger | n \right\rangle - \langle k | \hat{a}\hat{a} | n \rangle \right) \\ &= \frac{m\hbar\omega}{2} \left(2\sqrt{n} \left\langle k | \hat{a}^\dagger | n-1 \right\rangle + \langle k | n \rangle - \sqrt{n+1} \left\langle k | \hat{a}^\dagger | n+1 \right\rangle - \sqrt{n} \langle k | \hat{a} | n-1 \rangle \right) \\ &= \frac{m\hbar\omega}{2} \left(2n \langle k | n \rangle + \langle k | n \rangle - \sqrt{(n+2)(n+1)} \langle k | n+2 \rangle - \sqrt{n(n-1)} \langle k | n-2 \rangle \right), \end{aligned}$$

hence

$$\boxed{p^2_{kn} = \frac{m\hbar\omega}{2} \left[(2n+1)\delta_{kn} - \sqrt{(n+2)(n+1)}\delta_{k,n+2} - \sqrt{n(n-1)}\delta_{k,n-2} \right]}.$$

4. We can also use the result from (2) and write

$$\begin{aligned} p^2_{kn} &= p_{kl}p_{ln} \\ &= -\frac{m\hbar\omega}{2} \left(\sqrt{(l+1)}\delta_{k,l+1} - \sqrt{l}\delta_{k,l-1} \right) \left(\sqrt{(n+1)}\delta_{l,n+1} - \sqrt{n}\delta_{l,n-1} \right) \\ &= -\frac{m\hbar\omega}{2} \left[\sqrt{(l+1)(n+1)}\delta_{l,n+1}\delta_{k,l+1} - \sqrt{l(n+1)}\delta_{l,n+1}\delta_{k,l-1} - \sqrt{n(l+1)}\delta_{l,n-1}\delta_{k,l+1} + \sqrt{nl}\delta_{l,n-1}\delta_{k,l-1} \right] \\ &= -\frac{m\hbar\omega}{2} \left[\sqrt{(n+2)(n+1)}\delta_{k,n+2} - (n+1)\delta_{kn} - n\delta_{kn} + \sqrt{n(n-1)}\delta_{k,n-2} \right], \end{aligned}$$

thus

$$\boxed{p^2_{kn} = \frac{m\hbar\omega}{2} \left[(2n+1)\delta_{kn} - \sqrt{(n+2)(n+1)}\delta_{k,n+2} - \sqrt{n(n-1)}\delta_{k,n-2} \right]}.$$

5. Instead of using the eigenfunctions we will use the Dirac notation and write

$$\hat{x} = \sqrt{\frac{\hbar}{2m\omega}}\left(\hat{a}^{\dagger} + \hat{a}\right), \quad \hat{p} = i\sqrt{\frac{m\hbar\omega}{2}}\left(\hat{a}^{\dagger} - \hat{a}\right),$$

$$\hat{x}^2 = \frac{\hbar}{2m\omega}\left(\hat{a}^{\dagger}\hat{a}^{\dagger} + \hat{a}^{\dagger}\hat{a} + \hat{a}\hat{a}^{\dagger} + \hat{a}\hat{a}\right) = \frac{\hbar}{2m\omega}\left(\hat{a}^{\dagger}\hat{a}^{\dagger} + \hat{a}\hat{a} + 2\hat{a}^{\dagger}\hat{a} + 1\right),$$

and

$$\hat{p}^2 = 2m\hat{H} - (m\omega)^2\hat{x}^2 = \frac{m\hbar\omega}{2}\left(2\hat{a}^{\dagger}\hat{a} + 1 - \hat{a}^{\dagger}\hat{a}^{\dagger} - \hat{a}\hat{a}\right).$$

Thus, in the general case we have

$$\begin{aligned}\langle k|\hat{x}|n\rangle &= \sqrt{\frac{\hbar}{2m\omega}}\langle k|\left(\hat{a}^{\dagger} + \hat{a}\right)|n\rangle = \sqrt{\frac{\hbar}{2m\omega}}\left(\sqrt{n+1}\langle k|n+1\rangle + \sqrt{n}\langle k|n-1\rangle\right) \\ &= \sqrt{\frac{\hbar}{2m\omega}}\left(\sqrt{n+1}\delta_{k,n+1} + \sqrt{n}\delta_{k,n-1}\right), \\ \langle k|\hat{p}|n\rangle &= i\sqrt{\frac{\hbar m\omega}{2}}\langle k|\left(\hat{a}^{\dagger} - \hat{a}\right)|n\rangle = i\sqrt{\frac{\hbar m\omega}{2}}\left(\sqrt{n+1}\delta_{k,n+1} - \sqrt{n}\delta_{k,n-1}\right),\end{aligned}$$

and in the same manner

$$\begin{aligned}\langle k|\hat{x}^2|n\rangle &= \frac{\hbar}{2m\omega}\left(\sqrt{(n+1)(n+2)}\delta_{k,n+2} + \sqrt{n(n-1)}\delta_{k,n-2} + (2n+1)\delta_{k,n}\right), \\ \langle k|\hat{p}^2|n\rangle &= \frac{m\hbar\omega}{2}\left((2n+1)\delta_{k,n} - \sqrt{(n+1)(n+2)}\delta_{k,n+2} - \sqrt{n(n-1)}\delta_{k,n-2}\right).\end{aligned}$$

Therefore, for $n = k$ we get

$$\langle x\rangle_n = \langle p\rangle_n = 0,$$

and

$$\langle x^2\rangle_n = \sigma_x^2 = \frac{\hbar}{m\omega}\left(n + \frac{1}{2}\right) \quad \text{and} \quad \langle p^2\rangle = \sigma_p^2 = m\hbar\omega\left(n + \frac{1}{2}\right),$$

which yield the uncertainty relation

$$\boxed{\sigma_x\sigma_p = \frac{\hbar}{2}(2n+1)}.$$

The minimal value $\hbar/2$ corresponds to $n = 0$. The corresponding eigenfunction is the Gaussian

$$\boxed{\psi_0(x) = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} e^{-\frac{m\omega}{2\hbar}x^2}}.$$

6. The kinetic and potential energies are

$$\boxed{\langle T\rangle_n = \frac{\langle p^2\rangle_n}{2m} = \frac{\hbar\omega}{2}\left(n + \frac{1}{2}\right) = \frac{E_n}{2} \quad \text{and} \quad \langle V\rangle_n = \frac{1}{2}m\omega^2\langle x^2\rangle = \frac{\hbar\omega}{2}\left(n + \frac{1}{2}\right) = \frac{E_n}{2}}.$$

This is the same result one would've derived from the Virial theorem $2\langle T\rangle = n\langle V\rangle$, where $V \propto x^n$.

Question 3:

Among the stationary states of the harmonic oscillator only $n = 0$ hits the uncertainty limit ($\sigma_x\sigma_p = \hbar/2$). In general, $\sigma_x\sigma_p = (2n+1)/2$, as you found in the previous question. But certain linear combinations (known as *coherent states*) also minimize the uncertainty product. They are (as it turns out) eigenfunctions of the lowering operator (raising operator does not have eigenfunctions):

$$\hat{a}|\alpha\rangle = \alpha|\alpha\rangle,$$

where α is in general a complex number.

1. Calculate $\langle x\rangle$, $\langle p\rangle$, $\langle x^2\rangle$, $\langle p^2\rangle$ in the state $|\alpha\rangle$.
2. Find σ_x and σ_p. Show that $\sigma_x\sigma_p = \hbar/2$.
3. Like any other wave function, a coherent state can be expanded in terms of energy eigenstates:

$$|\alpha\rangle = \sum_{n=0}^{\infty} c_n |n\rangle .$$

   Show that the expansion coefficients are

$$c_n = \frac{\alpha^n}{\sqrt{n!}} c_0.$$

4. Show, by normalizing $|\alpha\rangle$, that

$$c_0 = e^{-\frac{|\alpha|^2}{2}}.$$

5. Show that $|\alpha(t)\rangle$ remains an eigenstate of $\hat{a}$, but the eigenvalue evolves in time:

$$\alpha(t) = e^{-i\omega t}\alpha.$$

   So a coherent state stays coherent, and continues to minimize the uncertainty product.
6. Based on your answers to (1), (2), and (5), find $\langle x\rangle(t)$ and $\sigma_x(t)$. It helps if you write the complex number α as

$$\alpha = C\sqrt{\frac{m\omega}{2\hbar}} e^{i\phi},$$

   for real numbers C and ϕ. *Comment*: In a sense, coherent states behave quasi-classically.
7. Is the ground state $|0\rangle$ itself a coherent state? If so, what is the eigenvalue?

Solution:

1. We can write $\hat{x}$ and $\hat{p}$ in terms of the ladder operators

$$\hat{x} = \sqrt{\frac{\hbar}{2m\omega}}\left(\hat{a}^\dagger + \hat{a}\right), \quad \hat{p} = i\sqrt{\frac{m\hbar\omega}{2}}\left(\hat{a}^\dagger - \hat{a}\right),$$

   then

$$\langle x\rangle = \sqrt{\frac{\hbar}{2m\omega}}\langle\alpha|\left(\hat{a}^\dagger + \hat{a}\right)|\alpha\rangle = \sqrt{\frac{\hbar}{2m\omega}}\left(\langle a\alpha|\alpha\rangle + \langle\alpha|a\alpha\rangle\right) = \boxed{\sqrt{\frac{\hbar}{2m\omega}}\left(\alpha^* + \alpha\right)},$$

$$\langle p\rangle = i\sqrt{\frac{m\hbar\omega}{2}}\langle\alpha|\left(\hat{a}^\dagger - \hat{a}\right)|\alpha\rangle = i\sqrt{\frac{m\hbar\omega}{2}}\left(\langle a\alpha|\alpha\rangle - \langle\alpha|a\alpha\rangle\right) = \boxed{i\sqrt{\frac{m\hbar\omega}{2}}\left(\alpha^* - \alpha\right)}.$$

   and

$$\begin{aligned}
\langle x^2\rangle &= \frac{\hbar}{2m\omega}\langle\alpha|\left(\hat{a}^\dagger\hat{a}^\dagger + \hat{a}^\dagger\hat{a} + \hat{a}\hat{a}^\dagger + \hat{a}\hat{a}\right)|\alpha\rangle \\
&= \frac{\hbar}{2m\omega}\langle\alpha|\left(\hat{a}^\dagger\hat{a}^\dagger + 2\hat{a}^\dagger\hat{a} + 1 + \hat{a}\hat{a}\right)|\alpha\rangle \\
&= \frac{\hbar}{2m\omega}\left(\alpha^{*2} + 2\alpha^*\alpha + 1 + \alpha^2\right) \\
&= \boxed{\frac{\hbar}{2m\omega}\left[1 + \left(\alpha^* + \alpha\right)^2\right]}, \\
\langle p^2\rangle &= -\frac{m\hbar\omega}{2}\langle\alpha|\left(\hat{a}^\dagger\hat{a}^\dagger - \hat{a}^\dagger\hat{a} - \hat{a}\hat{a}^\dagger + \hat{a}\hat{a}\right)|\alpha\rangle \\
&= -\frac{m\hbar\omega}{2}\langle\alpha|\left(\hat{a}^\dagger\hat{a}^\dagger - 2\hat{a}^\dagger\hat{a} - 1 + \hat{a}\hat{a}\right)|\alpha\rangle \\
&= -\frac{m\hbar\omega}{2}\left(\alpha^{*2} - 2\alpha^*\alpha - 1 + \alpha^2\right) \\
&= \boxed{\frac{m\hbar\omega}{2}\left[1 - \left(\alpha^* - \alpha\right)^2\right]}.
\end{aligned}$$

2. Using the results from (1) we have

$$\sigma_x^2 = \frac{\hbar}{2m\omega}\left[1 + (\alpha^* + \alpha)^2 - (\alpha^* + \alpha)^2\right] = \boxed{\frac{\hbar}{2m\omega}},$$
$$\sigma_p^2 = \frac{m\hbar\omega}{2}\left[1 - (\alpha^* - \alpha)^2 + (\alpha^* - \alpha)^2\right] = \boxed{\frac{m\hbar\omega}{2}},$$

thus $\boxed{\sigma_x \sigma_p = \hbar/2}$.

3. Taking the inner product with $|n\rangle$ we have

$$c_n = \langle n|\alpha\rangle = \left\langle 0\left|\frac{\hat{a}^n}{\sqrt{n!}}\right|\alpha\right\rangle = \frac{\alpha^n}{\sqrt{n!}}\langle 0|\alpha\rangle = \boxed{\frac{\alpha^n}{\sqrt{n!}}c_0}.$$

4. Using the representation

$$|\alpha\rangle = \sum_{n=0}^{\infty}\frac{\alpha^n}{\sqrt{n!}}c_0\,|n\rangle\,,$$

we require

$$\begin{aligned}
1 &= \langle\alpha|\alpha\rangle \\
&= \sum_{n,m}\langle m|\, c_0^*\frac{(\alpha^*)^m}{\sqrt{m!}}\frac{\alpha^n}{\sqrt{n!}}c_0\,|n\rangle \\
&= |c_0|^2\sum_{n,m}\frac{(\alpha^*)^m\alpha^n}{\sqrt{m!n!}}\langle m|n\rangle \\
&= |c_0|^2\sum_{n}\frac{|\alpha|^{2n}}{n!} \\
&= |c_0|^2 e^{|\alpha|^2},
\end{aligned}$$

thus

$$\boxed{c_0 = e^{-|\alpha|^2/2}}.$$

5. Tacking the time dependence factor we get

$$\begin{aligned}
|\alpha(t)\rangle &= \sum_{n=0} c_n e^{-iE_nt/\hbar}\hat{a}\,|n\rangle \\
&= \sum_n \frac{\alpha^n}{\sqrt{n!}}e^{-i\left(n+\frac{1}{2}\right)\omega t}e^{-\frac{|\alpha|^2}{2}}\,|n\rangle \\
&= e^{-i\omega t/2}\sum_n\frac{\left(\alpha e^{-i\omega t}\right)^n}{\sqrt{n!}}e^{-\frac{|\alpha|^2}{2}}\,|n\rangle\,,
\end{aligned}$$

thus, aside from the phase $e^{-i\omega t/2}$ (which does not affect its status as an eigenfunction of $\hat{a}$, or its eigenvalue), $|\alpha(t)\rangle$ is the same as $|\alpha\rangle$, only with an eigenvalue of $\alpha(t) = \alpha e^{-i\omega t}$.

6. Using the results from before and taking the hint, we have

$$\begin{aligned}
\langle x\rangle(t) &= \sqrt{\frac{\hbar}{2m\omega}}\left[\alpha^*(t) + \alpha(t)\right] \\
&= \sqrt{\frac{\hbar}{2m\omega}}\left[\alpha^* e^{i\omega t} + \alpha e^{-i\omega t}\right] \\
&= \sqrt{\frac{\hbar}{2m\omega}}\left[C\sqrt{\frac{m\omega}{2\hbar}}e^{-i\phi}e^{i\omega t} + C\sqrt{\frac{m\omega}{2\hbar}}e^{i\phi}e^{-i\omega t}\right] \\
&= \frac{C}{2}\left[e^{i(\omega t-\phi)} + e^{-i(\omega t-\phi)}\right] \\
&= \boxed{C\cos(\omega t - \phi)},
\end{aligned}$$

and

$$\boxed{\sigma_x = \sqrt{\frac{\hbar}{2m\omega}}}.$$

So in a coherent state the expectation value of the position oscillates, just like all eigenstates of the harmonic oscillator, while the wave packet maintains a constant width.

7. Acting with $\hat{a}$ on the ground state yields

$$\hat{a}\left|0\right\rangle = 0,$$

which means it can be interpreted as an eigenstate with eigenvalue $\boxed{\alpha = 0}$.

Question 4:

A polar representation of the creation and annihilation operators for a simple harmonic oscillator can be written as:

$$\hat{a} \equiv \sqrt{\hat{N}+1}e^{i\hat{\phi}} \quad \text{and} \quad \hat{a}^\dagger = e^{-i\hat{\phi}}\sqrt{\hat{N}+1},$$

where $\hat{N} \equiv \hat{a}^\dagger \hat{a}$ is called the *number operator;* both $\hat{N}$ and $\hat{\phi}$ are hermitian.

1. Starting with the commutation relation $\left[\hat{a}, \hat{a}^\dagger\right] = 1$, show that

$$\left[e^{i\hat{\phi}}, \hat{N}\right] = e^{i\hat{\phi}} \quad \text{and} \quad \left[e^{-i\hat{\phi}}, \hat{N}\right] = -e^{-i\hat{\phi}}.$$

Similarly, Show that

$$\left[\cos\hat{\phi}, \hat{N}\right] = i\sin\hat{\phi} \quad \text{and} \quad \left[\sin\hat{\phi}, \hat{N}\right] = -i\cos\hat{\phi}.$$

2. Calculate the matrix elements

$$\left\langle k|e^{\pm i\hat{\phi}}|n\right\rangle, \quad \left\langle k|\cos\hat{\phi}|n\right\rangle, \quad \left\langle k|\sin\hat{\phi}|n\right\rangle.$$

3. The generalized Heisenberg relation states that for any two operators $\hat{A}$ and $\hat{B}$

$$\sigma_A^2\sigma_B^2 \geq \frac{1}{4}\left|\left\langle\left[\hat{A}, \hat{B}\right]\right\rangle\right|^2.$$

Write down the Heisenberg uncertainty relation between the operators $\hat{N}$ and $\cos\hat{\phi}$. Compute the quantities involved for the state

$$\left|\psi\right\rangle = \left(1-|c|^2\right)^{1/2}\sum_{n=0}^{\infty} c^n\left|n\right\rangle,$$

where c is some complex parameter. Show that the resulting inequality is always true.

4. Consider a *coherent* state

$$\left|\alpha\right\rangle = e^{-\frac{|\alpha|^2}{2}}\sum_{n=0}^{\infty}\frac{\alpha^n}{\sqrt{n!}}\left|n\right\rangle.$$

Calculate the quantities σ_N^2, $\sigma_{\cos\phi}^2$ and $\left\langle\sin\hat{\phi}\right\rangle$ in this state. Show that the *number-phase* uncertainty inequality reduces to an equality in the limit of very large occupation numbers ($\alpha \to \infty$).
You may use the asymptotic formulas:

$$\sum_{n=0}^{\infty}\frac{|z|^{2n}}{n!\sqrt{n+1}} \approx \frac{e^{|z|^2}}{|z|}\left(1-\frac{1}{8|z|^2}+\cdots\right),$$

$$\sum_{n=0}^{\infty}\frac{|z|^{2n}}{n!\sqrt{(n+1)(n+2)}} \approx \frac{e^{|z|^2}}{|z|}\left(1-\frac{1}{2|z|^2}+\cdots\right).$$

Solution:

1. Starting from $\left[\hat{a}, \hat{a}^\dagger\right] = 1$ we get

$$\begin{aligned}
1 &= \left[\hat{a}, \hat{a}^\dagger\right] \\
&= \left[\sqrt{\hat{N}+1}e^{i\hat{\phi}}, e^{-i\hat{\phi}}\sqrt{\hat{N}+1}\right] \\
&= \sqrt{\hat{N}+1}e^{i\hat{\phi}}e^{-i\hat{\phi}}\sqrt{\hat{N}+1} - e^{-i\hat{\phi}}\sqrt{\hat{N}+1}\sqrt{\hat{N}+1}e^{i\hat{\phi}} \\
&= \hat{N} - e^{-i\hat{\phi}}\hat{N}e^{i\hat{\phi}},
\end{aligned}$$

where we used the hermiticity of $\hat{\phi}$ for $e^{-i\hat{\phi}} = \left(e^{i\hat{\phi}}\right)^\dagger$. Next we may act on this equation with either $e^{\pm i\hat{\phi}}$ from the right or left to get

$$\begin{aligned}
e^{-i\hat{\phi}} &= \hat{N}e^{-i\hat{\phi}} - e^{-i\hat{\phi}}\hat{N} = -\left[e^{-i\hat{\phi}}, \hat{N}\right], \\
e^{i\hat{\phi}} &= e^{i\hat{\phi}}\hat{N} - \hat{N}e^{i\hat{\phi}} = \left[e^{i\hat{\phi}}, \hat{N}\right],
\end{aligned}$$

thus

$$\boxed{\left[e^{\pm i\hat{\phi}}, \hat{N}\right] = \pm e^{\pm i\hat{\phi}}}.$$

We may use the Euler relation to write

$$\begin{aligned}
\left[\cos\hat{\phi}, \hat{N}\right] &= \frac{1}{2}\left(\left[e^{i\hat{\phi}}, \hat{N}\right] + \left[e^{-i\hat{\phi}}, \hat{N}\right]\right) = \frac{1}{2}\left(e^{i\hat{\phi}} - e^{-i\hat{\phi}}\right) = \boxed{i\sin\hat{\phi}}, \\
\left[\sin\hat{\phi}, \hat{N}\right] &= \frac{1}{2i}\left(\left[e^{i\hat{\phi}}, \hat{N}\right] - \left[e^{-i\hat{\phi}}, \hat{N}\right]\right) = \frac{1}{2i}\left(e^{i\hat{\phi}} + e^{-i\hat{\phi}}\right) = \boxed{-i\cos\hat{\phi}}.
\end{aligned}$$

2. The matrix elements of $e^{\pm i\hat{\phi}}$ are

$$\begin{aligned}
\left\langle k|e^{\pm i\hat{\phi}}|n\right\rangle &= \pm\left\langle k|\left[e^{\pm i\hat{\phi}}, \hat{N}\right]|n\right\rangle \\
&= \pm\left\langle k|e^{\pm i\hat{\phi}}\hat{N} - \hat{N}e^{\pm i\hat{\phi}}|n\right\rangle \\
&= \pm(n-k)\left\langle k|e^{\pm i\hat{\phi}}|n\right\rangle,
\end{aligned}$$

thus

$$(k - (n \mp 1))\left\langle k|e^{\pm i\hat{\phi}}|n\right\rangle = 0,$$

but $\left\langle k|e^{\pm i\hat{\phi}}|n\right\rangle \neq 0$, thus $k = n \mp 1$, meaning

$$\left\langle k|e^{\pm i\hat{\phi}}|n\right\rangle = c\delta_{k,n\mp 1}.$$

In order to find c we take the non-zero inner product

$$c = \left\langle n-1|e^{i\hat{\phi}}|n\right\rangle = \frac{1}{\sqrt{n}}\left\langle n-1|e^{i\hat{\phi}}\hat{a}^\dagger|n-1\right\rangle = \frac{1}{\sqrt{n}}\left\langle n-1|e^{i\hat{\phi}}e^{-i\hat{\phi}}\sqrt{\hat{N}+1}|n-1\right\rangle = \frac{1}{\sqrt{n}}\langle n-1|\sqrt{n}|n-1\rangle = 1.$$

Therefore

$$\boxed{\left\langle k|e^{\pm i\hat{\phi}}|n\right\rangle = \delta_{k,n\mp 1}}.$$

The rest are straightforward:

$$\begin{aligned}
\left\langle k|\cos\hat{\phi}|n\right\rangle &= \frac{1}{2}\left\langle k|e^{i\hat{\phi}} + e^{-i\hat{\phi}}|n\right\rangle = \boxed{\frac{1}{2}\left(\delta_{k,n-1} + \delta_{k,n+1}\right)}, \\
\left\langle k|\sin\hat{\phi}|n\right\rangle &= \frac{1}{2i}\left\langle k|e^{i\hat{\phi}} - e^{-i\hat{\phi}}|n\right\rangle = \boxed{\frac{1}{2i}\left(\delta_{k,n-1} - \delta_{k,n+1}\right)}.
\end{aligned}$$

3. We've already calculated the commutator in the right-hand-side $\left[\cos\hat{\phi}, \hat{N}\right] = i\sin\hat{\phi}$, thus, in order to check the inequality we have to calculate $\left\langle \hat{N}\right\rangle$, $\left\langle \hat{N}^2\right\rangle$, $\left\langle \cos\hat{\phi}\right\rangle$, $\left\langle \cos^2\hat{\phi}\right\rangle$ and $\left\langle \sin\hat{\phi}\right\rangle$. Note that

$$\hat{N}|n\rangle = \hat{a}^\dagger\hat{a}|n\rangle = \sqrt{n}\hat{a}^\dagger|n-1\rangle = n|n\rangle.$$

Thus:

$$\begin{aligned}
\langle N\rangle &= \left\langle \left(1-|c|^2\right)^{1/2}\sum_{k=0}^{\infty} c^k k\middle|\hat{N}\middle|\left(1-|c|^2\right)^{1/2}\sum_{n=0}^{\infty} c^n n\right\rangle \\
&= \left(1-|c|^2\right)\sum_{k,n=0}^{\infty}\left(c^*\right)^k c^n\left\langle k\middle|\hat{N}\middle|n\right\rangle \\
&= \left(1-|c|^2\right)\sum_{k,n=0}^{\infty}\left(c^*\right)^k c^n n\delta_{k,n} \\
&= \left(1-|c|^2\right)\sum_{n=0}^{\infty}|c|^{2n} n \\
&= \underline{\frac{|c|^2}{1-|c|^2}},
\end{aligned}$$

and in the same way

$$\left\langle N^2\right\rangle = \left(1-|c|^2\right)\sum_{n=0}^{\infty}|c|^{2n}n^2 = \underline{|c|^2\frac{1+|c|^2}{\left(1-|c|^2\right)^2}}.$$

For the rest, let us first consider

$$\begin{aligned}
\left\langle e^{\pm i\hat{\phi}}\right\rangle &= \left(1-|c|^2\right)\sum_{k,n=0}^{\infty}\left(c^*\right)^k c^n\left\langle k|e^{\pm i\hat{\phi}}|n\right\rangle \\
&= \left(1-|c|^2\right)\sum_{k,n=0}^{\infty}\left(c^*\right)^k c^n\delta_{k,n\mp 1} \\
&= \left(1-|c|^2\right)\sum_{k,n=0}^{\infty}|c|^{2n}c_{\mp} \\
&= c_{\mp},
\end{aligned}$$

where $c_{\mp}$ is c or c^*:

$$\left\langle e^{i\hat{\phi}}\right\rangle = c \quad \text{and} \quad \left\langle e^{-i\hat{\phi}}\right\rangle = c^*.$$

Therefore

$$\underline{\left\langle\cos\hat{\phi}\right\rangle = \frac{1}{2}\left(c+c^*\right) \quad \text{and} \quad \left\langle\sin\hat{\phi}\right\rangle = \frac{1}{2i}\left(c-c^*\right)},$$

and

$$
\begin{aligned}
\left\langle \cos^2 \hat{\phi} \right\rangle &= \left(1 - |c|^2\right) \sum_{k,n=0}^{\infty} (c^*)^k c^n \left\langle k \middle| \cos\hat{\phi} \cos\hat{\phi} \middle| n \right\rangle \\
&= \left(1 - |c|^2\right) \sum_{k,n,m=0}^{\infty} (c^*)^k c^n \left\langle k \middle| \cos\hat{\phi} \middle| m \right\rangle \left\langle m \middle| \cos\hat{\phi} \middle| n \right\rangle \\
&= \left(1 - |c|^2\right) \sum_{k,n,m=0}^{\infty} (c^*)^k c^n \frac{1}{4} \left(\delta_{k,m-1} + \delta_{k,m+1}\right) \left(\delta_{m,n-1} + \delta_{m,n+1}\right) \\
&= \left(1 - |c|^2\right) \sum_{k,n=0}^{\infty} (c^*)^k c^n \frac{1}{4} \left(\delta_{k,n-2} + \delta_{k,n} + \delta_{k,n} + \delta_{k,n+2}\right) \\
&= \left(1 - |c|^2\right) \sum_{n=0}^{\infty} \frac{1}{4} |c|^{2n} \left((c^*)^2 + 2 + c^2\right) \\
&= \underline{\frac{1}{2} + \frac{1}{4}\left(c^{*2} + c^2\right)}.
\end{aligned}
$$

Thus

$$
\begin{aligned}
\sigma_N^2 &= |c|^2 \frac{1 + |c|^2}{\left(1 - |c|^2\right)^2} - \frac{|c|^4}{\left(1 - |c|^2\right)^2} = \frac{|c|^2}{1 - |c|^2} \\
\sigma_{\cos\phi}^2 &= \frac{1}{2} + \frac{1}{4}\left(c^{*2} + c^2\right) - \frac{1}{4}\left(c + c^*\right)^2 = \frac{1}{2}\left(1 - |c|^2\right)
\end{aligned}
$$

$$
\sigma_{\cos\phi}^2 \sigma_N^2 \geq \frac{1}{4} \left| \left\langle \left[\hat{N}, \cos\hat{\phi}\right] \right\rangle \right|^2 \quad \rightarrow \quad \boxed{\frac{1}{2}|c|^2 \geq \frac{1}{16} \left|(c - c^*)\right|^2 = \frac{1}{8} \mathrm{Im}\,[c]^2},
$$

which holds true, since the extreme scenario is when c is imaginary, in which $|c| = \mathrm{Im}\,[c]$ and we have $1/2 > 1/8$.

4. Following the same method as before we calculate $\langle N \rangle$, $\langle N^2 \rangle$, $\left\langle \cos\hat{\phi} \right\rangle$, $\left\langle \cos^2\hat{\phi} \right\rangle$, $\left\langle \sin\hat{\phi} \right\rangle$ and $\left\langle \sin^2\hat{\phi} \right\rangle$:

$$
\begin{aligned}
\langle N \rangle &= \left\langle \alpha | \hat{N} | \alpha \right\rangle \\
&= e^{-|\alpha|^2} \sum_{k,n=0}^{\infty} \frac{(\alpha^*)^k \alpha^n}{\sqrt{k!n!}} \left\langle k | \hat{N} | n \right\rangle \\
&= e^{-|\alpha|^2} \sum_{k,n=0}^{\infty} \frac{(\alpha^*)^k \alpha^n}{\sqrt{k!n!}} n \langle k | n \rangle \\
&= e^{-|\alpha|^2} \sum_{n=0}^{\infty} \frac{|\alpha|^{2n}}{n!} n \\
&= e^{-|\alpha|^2} |\alpha|^2 \sum_{n=1}^{\infty} \frac{|\alpha|^{2(n-1)}}{(n-1)!} \\
&= e^{-|\alpha|^2} |\alpha|^2 \sum_{n=0}^{\infty} \frac{|\alpha|^{2n}}{n!} \\
&= \underline{|\alpha|^2},
\end{aligned}
$$

and in the same manner

$$\begin{aligned}\left\langle N^2\right\rangle &= \left\langle \alpha|\hat{N}^2|\alpha\right\rangle \\ &= e^{-|\alpha|^2}\sum_{n=1}^{\infty}\frac{|\alpha|^{2n}}{n!}n^2 \\ &= e^{-|\alpha|^2}|\alpha|^2\sum_{n=1}^{\infty}\frac{|\alpha|^{2(n-1)}}{(n-1)!}n \\ &= e^{-|\alpha|^2}|\alpha|^2\sum_{n=0}^{\infty}\frac{|\alpha|^{2n}}{n!}(n+1) \\ &= \underline{|\alpha|^2\left(|\alpha|^2+1\right)}.\end{aligned}$$

Therefore

$$\boxed{\sigma_N^2 = |\alpha|^2\left(|\alpha|^2+1\right)-|\alpha|^4 = |\alpha|^2}.$$

Just like before, let us start with

$$\begin{aligned}\left\langle e^{\pm i\hat{\phi}}\right\rangle &= e^{-|\alpha|^2}\sum_{k,n=0}^{\infty}\frac{(\alpha^*)^k\alpha^n}{\sqrt{k!n!}}\left\langle k|e^{\pm i\hat{\phi}}|n\right\rangle \\ &= e^{-|\alpha|^2}\sum_{k,n=0}^{\infty}\frac{(\alpha^*)^k\alpha^n}{\sqrt{k!n!}}\delta_{k,n\mp1} \\ &= e^{-|\alpha|^2}\sum_{n=0}^{\infty}\frac{|\alpha|^{2n}}{n!\sqrt{n+1}}\alpha_{\mp}\end{aligned}$$

such that

$$\left\langle e^{i\hat{\phi}}\right\rangle = \alpha e^{-|\alpha|^2}\sum_{n=0}^{\infty}\frac{|\alpha|^{2n}}{n!\sqrt{n+1}} \quad\text{and}\quad \left\langle e^{-i\hat{\phi}}\right\rangle = \alpha^* e^{-|\alpha|^2}\sum_{n=0}^{\infty}\frac{|\alpha|^{2n}}{n!\sqrt{n+1}},$$

Thus

$$\underline{\left\langle \cos\hat{\phi}\right\rangle = \frac{\alpha+\alpha^*}{2}e^{-|\alpha|^2}\sum_{n=0}^{\infty}\frac{|\alpha|^{2n}}{n!\sqrt{n+1}}} \quad\text{and}\quad \boxed{\left\langle \sin\hat{\phi}\right\rangle = \frac{\alpha-\alpha^*}{2}e^{-|\alpha|^2}\sum_{n=0}^{\infty}\frac{|\alpha|^{2n}}{n!\sqrt{n+1}}}.$$

Using these, we get

$$\begin{aligned}\left\langle \cos^2\hat{\phi}\right\rangle &= \left\langle \alpha|\cos\hat{\phi}\cos\hat{\phi}|\alpha\right\rangle \\ &= e^{-|\alpha|^2}\sum_{k,n=0}^{\infty}\frac{(\alpha^*)^k\alpha^n}{\sqrt{k!n!}}\left\langle k|\cos\hat{\phi}\cos\hat{\phi}|n\right\rangle \\ &= e^{-|\alpha|^2}\sum_{k,n,m=0}^{\infty}\frac{(\alpha^*)^k\alpha^n}{\sqrt{k!n!}}\left\langle k|\cos\hat{\phi}|m\right\rangle\left\langle m|\cos\hat{\phi}|n\right\rangle \\ &= e^{-|\alpha|^2}\sum_{k,n=0}^{\infty}\frac{(\alpha^*)^k\alpha^n}{\sqrt{k!n!}}\frac{1}{4}\left(\delta_{k,n-2}+2\delta_{k,n}+\delta_{k,n+2}\right) \\ &= \frac{1}{4}e^{-|\alpha|^2}\sum_{n=0}^{\infty}\left(\left(\alpha^2+(\alpha^*)^2\right)\frac{|\alpha|^{2n}}{\sqrt{(n+2)!n!}}+2\frac{|\alpha|^{2n}}{n!}\right) \\ &= \frac{1}{4}e^{-|\alpha|^2}\sum_{n=0}^{\infty}\left(\left(\alpha^2+(\alpha^*)^2\right)\frac{|\alpha|^{2n}}{n!\sqrt{(n+1)(n+2)}}+2\frac{|\alpha|^{2n}}{n!}\right) \\ &= \underline{\frac{1}{2}+\frac{\alpha^2+(\alpha^*)^2}{4}e^{-|\alpha|^2}\sum_{n=0}^{\infty}\frac{|\alpha|^{2n}}{n!\sqrt{(n+1)(n+2)}}},\end{aligned}$$

therefore

$$\boxed{\sigma^2_{\cos\phi} = \frac{1}{2} + \frac{\alpha^2 + (\alpha^*)^2}{4} e^{-|\alpha|^2} \sum_{n=0}^{\infty} \frac{|\alpha|^{2n}}{n!\sqrt{(n+1)(n+2)}} - \frac{(\alpha + \alpha^*)^2}{4} e^{-2|\alpha|^2} \left(\sum_{n=0}^{\infty} \frac{|\alpha|^{2n}}{n!\sqrt{n+1}} \right)^2}.$$

Recalling that $\sigma^2_{\cos\phi}\sigma^2_N \geq \frac{1}{4}\left|\left\langle \sin\hat{\phi} \right\rangle\right|^2$, we can plug it all in and find

$$|\alpha|^2 \left[\frac{1}{2} + \frac{\alpha^2 + (\alpha^*)^2}{4} e^{-|\alpha|^2} \sum_{n=0}^{\infty} \frac{|\alpha|^{2n}}{n!\sqrt{(n+1)(n+2)}} - \frac{(\alpha + \alpha^*)^2}{4} e^{-2|\alpha|^2} \left(\sum_{n=0}^{\infty} \frac{|\alpha|^{2n}}{n!\sqrt{n+1}} \right)^2 \right] \leq \frac{|\alpha - \alpha^*|^2}{16} e^{-2|\alpha|^2} \left(\sum_{n=0}^{\infty} \frac{|\alpha|^{2n}}{n!\sqrt{n}} \right.$$

taking $\alpha \to \infty$, and keeping the first order in the asymptotic formulas, we get

$$|\alpha|^2 \left[\frac{1}{2} + \frac{\alpha^2 + (\alpha^*)^2}{4|\alpha|^2} \left(1 - \frac{1}{2|\alpha|^2} + \mathcal{O}\left(|\alpha|^{-4}\right) \right) - \frac{(\alpha + \alpha^*)^2}{4|\alpha|^2} \left(1 - \frac{1}{8|\alpha|^2} + \mathcal{O}\left(|\alpha|^{-4}\right) \right)^2 \right] \leq \frac{|\alpha - \alpha^*|^2}{16|\alpha|^2} \left(1 + \mathcal{O}\left(|\alpha|^{-2}\right) \right)^2$$

$$4|\alpha|^2 \left[\cancel{2|\alpha|^2} + \left(\alpha^2 + (\alpha^*)^2\right) \left(\cancel{1} - \frac{1}{2|\alpha|^2} + \mathcal{O}\left(|\alpha|^{-4}\right) \right) - \left(2|\alpha|^2 + \alpha^2 + (\alpha^*)^2 \right) \left(\cancel{1} - \frac{1}{4|\alpha|^2} + \mathcal{O}\left(|\alpha|^{-4}\right) \right) \right] \leq |\alpha - \alpha^*|^2 \left(\right.$$

$$\boxed{2|\alpha|^2 - \left(\alpha^2 + (\alpha^*)^2\right) \leq |\alpha - \alpha^*|^2 + \mathcal{O}\left(|\alpha|^{-2}\right),}$$

and since $|\alpha - \alpha^*|^2 = 2|\alpha|^2 - \alpha^2 - (\alpha^*)^2$, we get an equality in this limit.