Class Exercise 2 - The Dirac delta function

The delta-function is a generalized function. It can be thought of as the limit of a series of functions which become increasingly tall and narrow, with the following defining properties:

1. $\delta(x) = \begin{cases} \infty & x = 0 \\ 0 & x \neq 0 \end{cases}$

2. $\int\limits_{-\infty}^{\infty} \delta(x)\, dx = 1$

3. $\int\limits_{-\infty}^{\infty} \delta(x - x_0) f(x)\, dx = f(x_0)$

Note that the integral property applies to any choice of bounds which contain the point $x = 0$. The generic form of this property could be written as

$$\int\limits_{x_0-\varepsilon}^{x_0+\varepsilon} \delta(x - x_0)\, dx = 1,$$

for any $\varepsilon > 0$.

The delta-function is used to sample quantities, such as charge density, for example a single point charge q located at x_0 has a charge density of

$$\rho(x) = q\delta(x - x_0).$$

Using the superposition principle, we can write the charge density of N such point charges. For example, in $3d$ we have

$$\rho(\vec{r}) = \sum_{i=1}^{N} q_i \delta^{(3)}(\vec{r} - \vec{r}_i).$$

There are many examples of functions with a limiting behavior of a delta-function:

- Lorentzian - $f(x) = \frac{1}{\pi}\frac{\epsilon}{x^2+\epsilon^2}$,
- Gaussian - $f(x) = \frac{1}{\epsilon\sqrt{\pi}} \exp(-x^2/\epsilon^2)$,

- Sinc - $f(x) = \frac{1}{\pi}\frac{\sin(x/\epsilon)}{x}$.

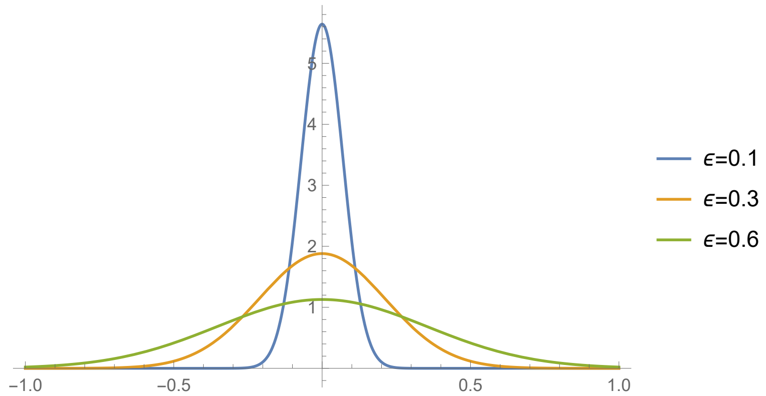

Fig. 1: The Gaussian $f(x) = \frac{1}{\epsilon\sqrt{\pi}}\exp\left(-x^2/\epsilon^2\right)$ for decreasingly small values of ϵ tends to a delta-function.

It is straightforward to check that taking $\epsilon \to 0$ gives a delta function in each of these cases. Some additional properties of the delta-function include

- $\delta(x) = \delta(-x)$,
- $\delta(\alpha x) = \frac{1}{|\alpha|\delta(x)}$
- $\delta(g(x)) = \sum_i \frac{\delta(x-x_i)}{|g'(x_i)|}$

Another useful function is the Heaviside function (a.k.a the "step" function), defined as

$$\Theta(x) = \begin{cases} 1, & x > 0, \\ 0, & x < 0. \end{cases}$$

It can be shown that

$$\Theta'(x) = \delta(x).$$

Problem 1

Prove the relation

$$\delta(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{d}k e^{ikx}. \tag{1}$$

Solution:

To perform the integration, we use the common trick of adding an ε contribution to the integral (which we will ultimately set to zero when we are done) in the following way:

$$\int_{-\infty}^{\infty} \mathrm{d}k e^{ikx} = \int_{-\infty}^{0} \mathrm{d}k e^{ikx} + \int_{0}^{\infty} \mathrm{d}k e^{ikx} = \lim_{\varepsilon\to 0^+}\left(\int_{-\infty}^{0} \mathrm{d}k e^{ik(x-i\varepsilon)} + \int_{0}^{\infty} \mathrm{d}k e^{ik(x+i\varepsilon)}\right). \tag{2}$$

The integration is carried before taking the limit,

$$\begin{aligned}
\int_{-\infty}^{\infty} \mathrm{d}k e^{ikx} &= \lim_{\varepsilon\to 0^+}\left(\frac{e^{ik(x-i\varepsilon)}}{i(x-i\varepsilon)}\Bigg|_{k=-\infty}^{0} + \frac{e^{ik(x+i\varepsilon)}}{i(x+i\varepsilon)}\Bigg|_{k=0}^{\infty}\right) \\
&= \lim_{\varepsilon\to 0^+}\left(\frac{1}{i(x-i\varepsilon)} - \frac{1}{i(x+i\varepsilon)} - \lim_{k\to\infty} \overbrace{e^{-k\varepsilon}}^{\text{tends to zero}} \left[\overbrace{\frac{e^{-ikx}}{i(x-i\varepsilon)} + \frac{e^{ikx}}{i(x+i\varepsilon)}}^{\text{oscillates in bounded region}}\right]\right) \\
&= \lim_{\varepsilon\to 0^+}\left(\frac{1}{i(x-i\varepsilon)} - \frac{1}{i(x+i\varepsilon)}\right) \\
&= \lim_{\varepsilon\to 0^+}\left(\frac{x+i\varepsilon}{i(x^2+\varepsilon^2)} - \frac{x-i\varepsilon}{i(x^2+\varepsilon^2)}\right) \\
&= \lim_{\varepsilon\to 0^+}\frac{2\varepsilon}{x^2+\varepsilon^2} = 2\pi\delta(x),
\end{aligned} \tag{3}$$

where in the last line we used the Lorentzian limit of the delta function that you saw in class.

Problem 2

Recall that in Cartesian coordinates,

$$\delta^{(D)}\left(\mathbf{x}-\mathbf{x}_0\right)=\prod_{i=1}^{D}\delta\left(x_i\right)=\delta\left(x_1\right)\delta\left(x_2\right)\cdots\delta\left(x_D\right). \tag{4}$$

Find the delta function $\delta^{(3)}\left(\mathbf{x}-\mathbf{x}_0\right)$ in spherical coordinates and cylindrical coordinates.

Solution

Recall the multidimensional property (or definition) of the delta function

$$\int\cdots\int \mathrm{d}^D x\delta^{(D)}\left(\mathbf{x}\right)=1. \tag{5}$$

Hence, in Cartesian coordinates we have

$$\int\cdots\int \mathrm{d}^D x\delta^{(D)}\left(\mathbf{x}\right)=\prod_{i=1}^{D}\int \mathrm{d}x_i\delta\left(x_i\right). \tag{6}$$

In 3d spherical coordinates, we know that the volume element is $r^2\sin\theta\mathrm{d}r\mathrm{d}\theta\mathrm{d}\varphi$, and we therefore have that

$$\int\mathrm{d}r\int\mathrm{d}\theta\int\mathrm{d}\varphi r^2\sin\theta\delta^{(3)}\left(\vec{r}-\vec{r}_0\right)=1. \tag{7}$$

Indeed, to obtain a form similar to (5) (and with proper units of density!), we need to have

$$r^2\sin\theta\delta^{(3)}\left(\vec{r}-\vec{r}_0\right)=\delta\left(r-r_0\right)\delta\left(\theta-\theta_0\right)\delta\left(\varphi-\varphi_0\right), \tag{8}$$

and thus in spherical coordinates,

$$\delta^{(3)}\left(\vec{r}-\vec{r}_0\right)=\frac{1}{r^2\sin\theta}\delta\left(r-r_0\right)\delta\left(\theta-\theta_0\right)\delta\left(\varphi-\varphi_0\right). \tag{9}$$

In cylindrical coordinates, the volume element is $r\mathrm{d}r\mathrm{d}\theta$, and we similarly find

$$\delta^{(3)}\left(\vec{r}-\vec{r}_0\right)=\frac{1}{r}\delta\left(r-r_0\right)\delta\left(\theta-\theta_0\right)\delta\left(z-z_0\right). \tag{10}$$

Problem 3

Express the following charge densities using δ functions in the appropriate coordinates:

1. N point charge with different charges q_i and positions z_i on the $\hat{\mathbf{z}}$ axis.
2. Total charge Q uniformly distributed over a sphere with radius R.
3. Thin disk of radius R and total charge Q, located in the $x-y$ plane.

Solution

1. The charges are distributed over the $\hat{\mathbf{z}}$ axis, and are thus positioned at $x=y=0$. We therefore write the charge density as

$$\rho\left(\vec{x}\right)=\sum_{i=1}^{N}q_i\delta\left(x\right)\delta\left(y\right)\delta\left(z-z_i\right). \tag{11}$$

We can check that the total charge is correct by integration,

$$\iiint_V \rho\left(\vec{x}\right)\mathrm{d}^3x=\iiint_V \mathrm{d}x\mathrm{d}y\mathrm{d}z\sum_{i=1}^{N}q_i\delta\left(x\right)\delta\left(y\right)\delta\left(z-z_i\right)=\sum_{i=1}^{N}q_i. \tag{12}$$

2. We work in spherical coordinates. The surface charge density is

$$\sigma=\frac{Q}{4\pi R^2}, \tag{13}$$

concentrated entirely on the shell at $r=R$. We therefore have

$$\begin{aligned}\rho\left(\vec{r}\right)&=\int \mathrm{d}q\delta^{(3)}\left(\vec{x}-\vec{x}'\right)\\&=\sigma\int_0^{\pi}\mathrm{d}\theta'\int_0^{2\pi}\mathrm{d}\varphi' R^2\sin\theta'\frac{\delta\left(r-R\right)\delta\left(\theta-\theta'\right)\delta\left(\varphi-\varphi'\right)}{r^2\sin\theta}\\&=\sigma\int_0^{\pi}\mathrm{d}\theta'\int_0^{2\pi}\mathrm{d}\varphi'\delta\left(r-R\right)\delta\left(\theta-\theta'\right)\delta\left(\varphi-\varphi'\right)\\&=\sigma\delta\left(r-R\right)\\&=\frac{Q}{4\pi R^2}\delta\left(r-R\right).\end{aligned} \tag{14}$$

3. The surface charge density is

$$\sigma=\frac{Q}{\pi R^2}. \tag{15}$$

We work in cylindrical coordinates and integrate,

$$\begin{aligned}\rho\left(\vec{r}\right) &= \int \mathrm{d}q\delta^{(3)}\left(\vec{x}-\vec{x}'\right)\\ &= \sigma\int\limits_0^{2\pi}\mathrm{d}\theta'\int\limits_0^R \mathrm{d}r'r'\frac{\delta\left(r-r'\right)\delta\left(\theta-\theta'\right)\delta\left(z\right)}{r}\\ &= \sigma\delta\left(z\right)\int\limits_0^R \mathrm{d}r'\delta\left(r-r'\right).\end{aligned} \tag{16}$$

We recall that

$$\frac{\partial}{\partial r}\Theta\left(r-r'\right) = -\frac{\partial}{\partial r'}\Theta\left(r-r'\right) = \delta\left(r-r'\right), \tag{17}$$

which means we can write

$$\rho\left(\vec{r}\right) = -\sigma\delta\left(z\right)\int\limits_0^R \mathrm{d}r'\frac{\partial}{\partial r'}\Theta\left(r-r'\right) = \sigma\delta\left(z\right)\left[\Theta\left(r\right)-\Theta\left(r-R\right)\right]. \tag{18}$$

Note that

$$\Theta\left(r\right)-\Theta\left(r-R\right) = \Theta\left(R-r\right) = \begin{cases}1 & 0<r<R\\ 0 & r>R\end{cases}, \tag{19}$$

and therefore we find the charge density

$$\rho\left(\vec{r}\right) = \sigma\delta\left(z\right)\Theta\left(R-r\right) = \frac{Q}{\pi R^2}\delta\left(z\right)\Theta\left(R-r\right). \tag{20}$$

Problem 4

A thin ring of radius R is homogeneously charged with charge density λ. Find the potential on the symmetry axis of the ring, far away from the ring.

Solution

The solution to the Laplace equation is

$$\varphi\left(\vec{r}\right)=\iiint\frac{\rho\left(\vec{r}'\right)\mathrm{d}\vec{r}'}{\left|\vec{r}-\vec{r}'\right|}. \tag{21}$$

In our case, assuming the ring is located at $z=0$, we have the charge density

$$\rho\left(\vec{r}'\right)=\lambda\delta\left(r'-R\right)\delta\left(z'\right). \tag{22}$$

The chosen point of reference is somewhere on the $\hat{\mathbf{z}}$ axis, and therefore $\vec{r}=z\hat{\mathbf{z}}$, and the location of the charge is $\vec{r}'=R\hat{\mathbf{r}}$. Our integral is then

$$\begin{aligned}\varphi\left(\vec{r}\right)&=\lambda\int\mathrm{d}r'\int\mathrm{d}\theta'\int\mathrm{d}z'\frac{\delta\left(r'-R\right)\delta\left(z'\right)r'}{\left|z\hat{\mathbf{z}}-R\hat{\mathbf{r}}\right|}\\&=\lambda\iiint\frac{\delta\left(r'-R\right)\delta\left(z'\right)r'\mathrm{d}r'\mathrm{d}\theta'\mathrm{d}z'}{\sqrt{z^2+R^2}}\\&=\lambda\int_0^{2\pi}\frac{R\mathrm{d}\theta'}{\sqrt{z^2+R^2}}=\frac{2\pi R\lambda}{\sqrt{z^2+R^2}}.\end{aligned} \tag{23}$$

Far away from the ring we have $z\gg R$, and thus

$$\varphi\left(z\right)=\frac{2\pi R\lambda}{z\sqrt{1+\left(\frac{R}{z}\right)^2}}\underset{z\gg R}{\approx}\frac{2\pi R\lambda}{z}\left(1-\frac{1}{2}\left(\frac{R}{z}\right)^2\right). \tag{24}$$

Problem 5

Solve the 1d equation

$$\frac{d^2}{dx^2}G(x,x') = \delta(x-x'), \tag{25}$$

given the initial conditions $G(x=0,x') = G(x=L,x') = 0$.

Solution

First, we note that for $x \neq x'$,

$$\frac{d^2}{dx^2}G(x,x') = 0. \tag{26}$$

We might be tempted to say that the general solution is a simple linear function,

$$G(x,x') = Ax + B. \tag{27}$$

Consider the derivative of this solution,

$$\frac{d}{dx}G(x,x') = A. \tag{28}$$

We will show that this does not hold for all points x:

Integrating equation (25) from $x = x' - \epsilon$ to $x = x' + \epsilon$,

$$\int_{x'-\epsilon}^{x'+\epsilon} \frac{d^2}{dx^2}G(x,x')\,\mathrm{d}x = \int_{x'-\epsilon}^{x'+\epsilon} \delta(x-x')\,\mathrm{d}x, \tag{29}$$

we find

$$\left.\frac{dG}{dx}\right|_{x'+\epsilon} - \left.\frac{dG}{dx}\right|_{x'-\epsilon} = 1. \tag{30}$$

Therefore, equation (27) cannot be true for all x. However, we can split the solution into a pair of different linear solutions,

$$G(x,x') = \begin{cases} Ax + B, & 0 < x < x' < L, \\ Cx + D, & 0 < x' < x < L, \end{cases} \tag{31}$$

such that these satisfy the following conditions:

$$\overbrace{G(x = x' - \epsilon, x') = G(x = x' + \epsilon, x')}^{\text{Continuity}}, \tag{32}$$

$$\overbrace{\left.\frac{dG}{dx}\right|_{x'+\epsilon} - \left.\frac{dG}{dx}\right|_{x'-\epsilon} = 1,}^{\text{Jump in 1st derivative}} \tag{33}$$

$$\overbrace{G\left(x=0,x'\right)=0,}^{\text{Boundary condition}} \tag{34}$$

$$\overbrace{G\left(x=L,x'\right)=0.}^{\text{Boundary condition}} \tag{35}$$

From equation (34) we find $B = 0$, and from equation (35) we find

$$CL + D = 0. \tag{36}$$

So far, we find that the general solution takes the form

$$G\left(x,x'\right) = \begin{cases} Ax, & 0 < x < x' < L, \\ C\left(x-L\right), & 0 < x' < x < L. \end{cases} \tag{37}$$

We use the jump in the derivative (equation (33)) to determine that

$$C - A = 1, \tag{38}$$

and finally, invoke the continuity condition (32) as $\epsilon \to 0$, replace A by $C-1$ and obtain

$$\begin{aligned} \left(C-1\right)x' &= C\left(x'-L\right) \\ \implies C &= \frac{x'}{L}, \qquad (39) \\ \implies A &= \frac{x'-L}{L}. \qquad (40) \end{aligned}$$

The complete solution is then

$$G\left(x,x'\right) = \begin{cases} \left(\frac{x'-L}{L}\right)x, & 0 < x < x' < L, \\ \frac{x'}{L}\left(x-L\right), & 0 < x' < x < L. \end{cases} \tag{41}$$

Problem 6

Given the electric potential function

$$\varphi(\vec{r}) = \frac{1}{r}, \tag{42}$$

find the electric field and the charge density.

Solution

The potential only depends on r (and not on the angular coordinates), so we choose to solve this problem in spherical coordinates.

We begin by calculating the electric field. Recall from Physics 2 that

$$\vec{E}(\vec{r}) = -\vec{\nabla}\varphi(\vec{r}). \tag{43}$$

Since $\varphi = \varphi(r)$, in spherical coordinates we have

$$\vec{E}(\vec{r}) = -\frac{\partial \varphi(\vec{r})}{\partial r}\hat{\mathbf{r}} = \frac{\hat{\mathbf{r}}}{r^2}. \tag{44}$$

For the field density $\rho(\vec{r})$, we take the divergence of $\vec{E}$ (which is also the Laplacian of φ), and use the relation

$$\vec{\nabla} \cdot \vec{E} = 4\pi\rho. \tag{45}$$

Again, $\vec{E} = \vec{E}(r)$ and so we have

$$4\pi\rho(\vec{r}) = \frac{1}{r^2}\frac{\partial(r^2 E_r)}{\partial r} = \frac{1}{r^2}\frac{\partial(r^2/r^2)}{\partial r} = 0. \tag{46}$$

It appears that the divergence of the field is everywhere zero! But that is impossible - since we find that by applying Gauss' law to an integral of ball of radius r around the origin,

$$\iiint\limits_V \vec{\nabla} \cdot \vec{E} \mathrm{d}^3 r = \oiint\limits_{\partial V} \vec{E} \cdot \hat{\mathbf{r}} \mathrm{d}S = \oiint\limits_{\partial V} \left(\frac{\hat{\mathbf{r}}}{r^2}\right) \cdot \left(r^2 \mathrm{d}\Omega \hat{\mathbf{r}}\right) = \oiint\limits_{\partial V} \mathrm{d}\Omega = 4\pi, \tag{47}$$

where $\mathrm{d}\Omega = \sin\theta \mathrm{d}\theta \mathrm{d}\phi$ is the angular area element.

We must be careful in our treatment of the case $r = 0$:

$$4\pi\rho(\vec{r}) = \begin{cases} 0, & r \neq 0, \\ \lim_{r\to 0} \frac{1}{r^2}\frac{\partial\left(r^2/r^2\right)}{\partial r} = \infty, & r = 0. \end{cases}$$

Looks familiar? This form is precisely a delta function $\delta^{(3)}(\vec{r})$! Hence, the actual solution for the field density is

$$\rho\left(\vec{r}\right)=\frac{1}{4\pi}\delta^{(3)}\left(\vec{r}\right), \tag{48}$$

which is indeed in agreement with Gauss' law.

Remark We have proven a very important relation,

$$\nabla^{2}\left(\frac{1}{r}\right)=-4\pi\delta^{(3)}\left(\vec{r}\right). \tag{49}$$

Note that the origin can be displaced to give the general identity

$$\nabla^{2}\left(\frac{1}{\left|\vec{r}-\vec{r}_{0}\right|}\right)=-4\pi\delta^{(3)}\left(\vec{r}-\vec{r}_{0}\right). \tag{50}$$