Class Exercise 1 - Vector Analysis

A vector is a d-dimensional object, independent of the choice of coordinates.

In terms of a coordinate basis, e.g. some 3d basis $\hat{\mathbf{e}}_1, \hat{\mathbf{e}}_2, \hat{\mathbf{e}}_3$, we can represent the vector $\vec{V}$ in the following equivalent forms:

$$\vec{V} = (V_1, V_2, V_3) = V_1\hat{\mathbf{e}}_1 + V_2\hat{\mathbf{e}}_2 + V_3\hat{\mathbf{e}}_3 = \sum_{i=1}^{3} V_i\hat{\mathbf{e}}_i. \tag{1}$$

The last form is often written in shorthand notation ("Einstein summation notation") as $V_i\hat{\mathbf{e}}_i$, where *summation is implied by the repeating of an index*, for example,

$$V_iV_i = V_1V_1 + V_2V_2 + V_3V_3 = V^2. \tag{2}$$

The components V_i of a vector are often functions of the position coordinates, for example the Cartesian x, y, z. They may also be functions of the time t. Therefore, derivatives of vectors will have significant physical interpretations.

We define the derivative operator $\vec{\nabla} = \frac{\partial}{\partial x_i}\hat{\mathbf{e}}_i = \left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_3}\right)$. Its operations on vectors and scalar functions (in Cartesian coordinates) are:

$$\text{Gradient} \quad \vec{\nabla}\phi = \frac{\partial}{\partial x_i}\phi = \left(\frac{\partial}{\partial x}\phi, \frac{\partial}{\partial y}\phi, \frac{\partial}{\partial z}\phi\right),$$

$$\text{Divergence } \vec{\nabla}\cdot\vec{V} = \frac{\partial}{\partial x_i}V_i = \frac{\partial}{\partial x}V_x + \frac{\partial}{\partial y}V_y + \frac{\partial}{\partial z}V_z,$$

$$\text{Curl} \quad \vec{\nabla}\times\vec{V} = \varepsilon_{ijk}\frac{\partial}{\partial x_j}V_k = \left(\frac{\partial}{\partial y}V_z - \frac{\partial}{\partial z}V_y, \frac{\partial}{\partial z}V_x - \frac{\partial}{\partial x}V_z, \frac{\partial}{\partial x}V_y - \frac{\partial}{\partial y}V_x\right)$$

$$\text{Laplacian} \qquad \nabla^2\phi = \partial_i\partial_i\phi = \tfrac{\partial^2}{\partial x^2}\phi + \tfrac{\partial^2}{\partial y^2}\phi + \tfrac{\partial^2}{\partial z^2}\phi,$$

where

$$\varepsilon_{ijk} = \begin{cases} +1 & \text{for cyclic permutations of } 1,2,3, \\ 0 & \text{for repeated indices, e.g. } 1,1,2, \\ -1 & \text{for anti-cyclic permutations of } 1,2,3. \end{cases}$$

The cyclic permutations are $(1,2,3)\,,(2,3,1),\ (3,1,2)$, and the anti-cyclic permutations are $(1,3,2)\,,(2,1,3),\ (3,2,1)$.

During the course we will use spherical and cylindrical coordinates quite often, in which the derivative operations take less intuitive forms:

Cylindrical coordinates:

$$\vec{\nabla} f = \tfrac{\partial f}{\partial \rho}\hat{\boldsymbol{\rho}} + \tfrac{1}{\rho}\tfrac{\partial f}{\partial \varphi}\hat{\boldsymbol{\varphi}} + \tfrac{\partial f}{\partial z}\hat{\mathbf{z}},$$

$$\vec{\nabla}\cdot\vec{V} = \tfrac{1}{\rho}\tfrac{\partial(\rho V_\rho)}{\partial \rho} + \tfrac{1}{\rho}\tfrac{\partial V_\varphi}{\partial \varphi} + \tfrac{\partial V_z}{\partial z},$$

$$\vec{\nabla}\times\vec{V} = \left(\tfrac{1}{\rho}\tfrac{\partial V_z}{\partial \varphi} - \tfrac{\partial V_\varphi}{\partial z}\right)\hat{\boldsymbol{\rho}} + \left(\tfrac{\partial V_\rho}{\partial z} - \tfrac{\partial V_z}{\partial \rho}\right)\hat{\boldsymbol{\varphi}} + \tfrac{1}{\rho}\left(\tfrac{\partial(\rho V_\varphi)}{\partial \rho} - \tfrac{\partial V_\rho}{\partial \varphi}\right)\hat{\mathbf{z}},$$

$$\nabla^2 f = \tfrac{1}{\rho}\tfrac{\partial}{\partial \rho}\left(\rho\tfrac{\partial f}{\partial \rho}\right) + \tfrac{1}{\rho^2}\tfrac{\partial^2 f}{\partial \varphi^2} + \tfrac{\partial^2 f}{\partial z^2}.$$

Spherical Coordinates:

$$\vec{\nabla} f = \tfrac{\partial f}{\partial r}\hat{\mathbf{r}} + \tfrac{1}{r}\tfrac{\partial f}{\partial \theta}\hat{\boldsymbol{\theta}} + \tfrac{1}{r\sin\theta}\tfrac{\partial f}{\partial \varphi}\hat{\boldsymbol{\varphi}},$$

$$\vec{\nabla}\cdot\vec{V} = \frac{1}{r^2}\frac{\partial\left(r^2 V_r\right)}{\partial r} + \frac{1}{r\sinh\theta}\frac{\partial\left(V_\theta\sin\theta\right)}{\partial\theta} + \frac{1}{r\sin\theta}\frac{\partial V_\varphi}{\partial\varphi},$$

$$\vec{\nabla}\times\vec{V} = \frac{1}{r\sin\theta}\left(\frac{\partial\left(V_\varphi\sin\theta\right)}{\partial\theta} - \frac{\partial V_\theta}{\partial\varphi}\right)\hat{\mathbf{r}} + \frac{1}{r}\left(\frac{1}{\sin\theta}\frac{\partial V_r}{\partial\varphi} - \frac{\partial\left(rV_\varphi\right)}{\partial r}\right)\hat{\boldsymbol{\theta}} + \frac{1}{r}\left(\frac{\partial\left(rV_\theta\right)}{\partial r} - \frac{\partial V_r}{\partial\theta}\right)\hat{\boldsymbol{\varphi}},$$

$$\nabla^2 f = \tfrac{1}{r^2}\tfrac{\partial}{\partial r}\left(r^2\tfrac{\partial f}{\partial r}\right) + \tfrac{1}{r^2\sin\theta}\tfrac{\partial}{\partial\theta}\left(\sin\theta\tfrac{\partial f}{\partial\theta}\right) + \tfrac{1}{r^2\sin^2\theta}\tfrac{\partial^2 f}{\partial\varphi^2}.$$

Problem 1

1. Prove the identity
$$\vec{\nabla} \times \left(\vec{\nabla} \times \vec{V}\right) = \vec{\nabla}\left(\vec{\nabla} \cdot \vec{V}\right) - \nabla^2 \vec{V}, \tag{3}$$
without the use of specific coordinates.

2. Find the explicit expression for $\nabla^2 \vec{V}$ in Cartesian coordinates.

Solution:

1. We note that for any three vectors $\vec{A}, \vec{B}, \vec{C}$,
$$\vec{A} \times \left(\vec{B} \times \vec{C}\right) = \vec{B}\left(\vec{A} \cdot \vec{C}\right) - \vec{C}\left(\vec{A} \cdot \vec{B}\right). \tag{4}$$
Naively inserting $\vec{A} = \vec{\nabla}, \vec{B} = \vec{\nabla}, \vec{C} = \vec{V}$ into this identity gives
$$\vec{\nabla} \times \left(\vec{B} \times \vec{C}\right) = \vec{\nabla}\left(\vec{\nabla} \cdot \vec{V}\right) - \vec{V}\left(\vec{\nabla} \cdot \vec{\nabla}\right). \tag{5}$$
However, the second term is **incorrect**! $\vec{\nabla}$ is an *operator*, which means the order of derivatives is critical and may not be changed. Maintaining the order, the correct expression is:
$$\vec{\nabla} \times \left(\vec{\nabla} \times \vec{V}\right) = \vec{\nabla}\left(\vec{\nabla} \cdot \vec{V}\right) - \left(\vec{\nabla} \cdot \vec{\nabla}\right)\vec{V}. \tag{6}$$
Notice that $\vec{\nabla} \cdot \vec{\nabla} = \partial_i \partial_i = \nabla^2$, and therefore we have come to the desired result,
$$\vec{\nabla} \times \left(\vec{\nabla} \times \vec{V}\right) = \vec{\nabla}\left(\vec{\nabla} \cdot \vec{V}\right) - \nabla^2 \vec{V}. \tag{7}$$

2. Using index notation will simplify the computation in this case (and in many others). First, we rearrange the identity of the Laplacian,
$$\nabla^2 \vec{V} = \vec{\nabla}\left(\vec{\nabla} \cdot \vec{V}\right) - \vec{\nabla} \times \left(\vec{\nabla} \times \vec{V}\right). \tag{8}$$
Let i be either the x, y or z component. Then equation (8) can be written separately for each component,
$$\left(\nabla^2 \vec{V}\right)_i = \left(\vec{\nabla}\left(\vec{\nabla} \cdot \vec{V}\right)\right)_i - \left(\vec{\nabla} \times \left(\vec{\nabla} \times \vec{V}\right)\right)_i. \tag{9}$$
The first term in index notation is
$$\left(\vec{\nabla}\left(\vec{\nabla} \cdot \vec{V}\right)\right)_i = \frac{\partial}{\partial x_i}\left(\vec{\nabla} \cdot \vec{V}\right) = \frac{\partial}{\partial x_i}\frac{\partial}{\partial x_j} V_j. \tag{10}$$

The second term is a bit more tricky:

$$\left(\vec{\nabla}\times\left(\vec{\nabla}\times\vec{V}\right)\right)_i = \varepsilon_{ijk}\frac{\partial}{\partial x_j}\left(\vec{\nabla}\times\vec{V}\right)_k = \varepsilon_{ijk}\frac{\partial}{\partial x_j}\left(\varepsilon_{k\ell m}\frac{\partial}{\partial x_\ell}V_m\right) = \varepsilon_{ijk}\varepsilon_{k\ell m}\frac{\partial}{\partial x_j}\frac{\partial}{\partial x_\ell}V_m. \tag{11}$$

Note an important identity of the Levi-Civita symbol:

$$\varepsilon_{ijk}\varepsilon_{i\ell m} = \delta_{j\ell}\delta_{km} - \delta_{jm}\delta_{k\ell}, \tag{12}$$

and we also recall that $\delta_{ij}V_j = V_i$. Using this identity and the cyclic properties of the Levi-Civita symbol, we have

$$\varepsilon_{ijk}\varepsilon_{k\ell m} = \varepsilon_{kij}\varepsilon_{k\ell m} = \delta_{i\ell}\delta_{jm} - \delta_{im}\delta_{j\ell}, \tag{13}$$

and therefore

$$\left(\vec{\nabla}\times\left(\vec{\nabla}\times\vec{V}\right)\right)_i = \left(\delta_{i\ell}\delta_{jm} - \delta_{im}\delta_{j\ell}\right)\frac{\partial}{\partial x_j}\frac{\partial}{\partial x_\ell}V_m = \frac{\partial}{\partial x_j}\frac{\partial}{\partial x_i}V_j - \frac{\partial}{\partial x_j}\frac{\partial}{\partial x_j}V_i. \tag{14}$$

Note that in Cartesian coordinates the derivatives commute

$$\frac{\partial}{\partial x_j}\frac{\partial}{\partial x_i}V_j = \frac{\partial}{\partial x_i}\left(\frac{\partial}{\partial x_j}V_j\right), \tag{15}$$

and so we may write

$$\left(\vec{\nabla}\times\left(\vec{\nabla}\times\vec{V}\right)\right)_i = \frac{\partial}{\partial x_i}\frac{\partial}{\partial x_j}V_j - \frac{\partial^2}{\partial x_j^2}V_i. \tag{16}$$

Finally, we add the two contributions and obtain

$$\left(\nabla^2\vec{V}\right)_i = \frac{\partial}{\partial x_i}\frac{\partial}{\partial x_j}V_j - \frac{\partial}{\partial x_i}\frac{\partial}{\partial x_j}V_j + \frac{\partial^2}{\partial x_j^2}V_i = \frac{\partial^2}{\partial x_j^2}V_i, \tag{17}$$

and therefore

$$\nabla^2\vec{V} = \left(\nabla^2 V_x, \nabla^2 V_y, \nabla^2 V_z\right). \tag{18}$$

This simple result is unique to Cartesian coordinates!

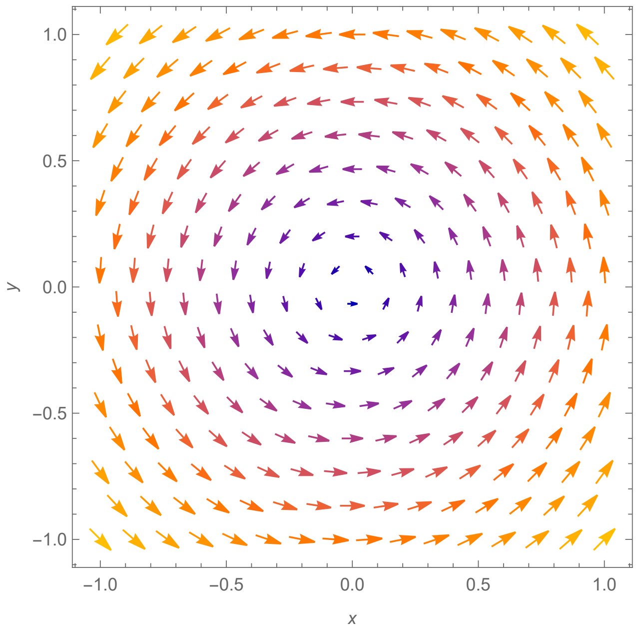

Fig. 1: Field lines of $\vec{G}(\vec{r})$ in any $z = \text{Const.}$ plane.

Problem 2

Given the vector function $\vec{G}(\vec{r}) = \hat{\mathbf{z}} \times \vec{r}$, where $\vec{r} = x\hat{\mathbf{x}} + y\hat{\mathbf{y}} + z\hat{\mathbf{z}}$:

1. Draw a rough sketch of the field lines.
2. Calculate $\oint\limits_C \vec{G} \cdot \mathrm{d}\vec{\ell}$ where $C = \{z = 0; x^2 + y^2 = 1\}$.

Solution

1. We work in cylindrical coordinates for convenience, where

$$\vec{r} = r\hat{\mathbf{r}} + z\hat{\mathbf{z}} = r\cos\theta\hat{\mathbf{x}} + r\sin\theta\hat{\mathbf{y}} + z\hat{\mathbf{z}}. \tag{19}$$

In cylindrical coordinates the radial unit vector is $\hat{r} = \cos\theta\hat{x} + \sin\theta\hat{y}$, and therefore we have that

$$\vec{G}(\vec{r}) = \hat{\mathbf{z}} \times \vec{r} = \hat{\mathbf{z}} \times (r\hat{\mathbf{r}} + z\hat{\mathbf{z}}) = r\hat{\boldsymbol{\theta}}, \tag{20}$$

where we have used that $\hat{\mathbf{z}} \times \hat{\mathbf{z}} = 0$ and that $\hat{\mathbf{z}} \times \hat{\mathbf{r}} = \hat{\boldsymbol{\theta}}$. Therefore, the field lines of the vector field $\vec{G}(\vec{r})$ draw circles in the $x - y$ plane around the origin $(x = y = 0)$ (figure 1).

2. In our chosen cylindrical coordinates, the curve C lies in the $z = 0$ plane on a circle of radius $r = \sqrt{x^2 + y^2} = 1$. We therefore carry the integration over the entire range of

the θ coordinate at a fixed r:

$$\begin{aligned}\oint_C \vec{G}\cdot \mathrm{d}\vec{\ell} &= \int_0^{2\pi} \vec{G}\left(r=1\right)\cdot \hat{\boldsymbol{\theta}}\mathrm{d}\theta \\ &= \int_0^{2\pi} \cancelto{1}{r}\hat{\boldsymbol{\theta}}\cdot \hat{\boldsymbol{\theta}}\mathrm{d}\theta \\ &= 2\pi. \end{aligned} \tag{21}$$

Another way to obtain this result is through the Stokes theorem:

$$\oint_C \vec{G}\cdot \mathrm{d}\vec{\ell} = \int_A \left(\vec{\nabla}\times\vec{G}\right)\cdot \mathrm{d}\vec{A}, \tag{22}$$

where A is the area of the domain enclosed by C. In cylindrical coordinates,

$$\vec{\nabla}\times\vec{G} = \left(\frac{1}{r}\frac{\partial G_z}{\partial\theta} - \frac{\partial G_\theta}{\partial z}\right)\hat{\mathbf{r}} + \left(\frac{\partial G_r}{\partial z} - \frac{\partial G_z}{\partial r}\right)\hat{\boldsymbol{\theta}} + \frac{1}{r}\left[\frac{\partial}{\partial r}\left(rG_\theta\right) - \frac{\partial}{\partial\theta}G_r\right]\hat{\mathbf{z}}. \tag{23}$$

Given that $\vec{G} = r\hat{\boldsymbol{\theta}}$, we find that

$$\vec{\nabla}\times\vec{G} = \frac{1}{r}\frac{\partial}{\partial r}\left(rG_\theta\right)\hat{\mathbf{z}} = 2\hat{\mathbf{z}}. \tag{24}$$

Therefore, since $d\vec{A} = dA\hat{\mathbf{z}}$, we find

$$\int_A 2\hat{\mathbf{z}}\cdot\hat{\mathbf{z}}\mathrm{d}A = 2A = 2\left(\text{Area of unit circle}\right) = 2\cdot\pi. \tag{25}$$

Note that we can also explicitly integrate over the area of the $r = 1$ circle by choosing a radial coordinate $0 \leq r \leq 1$ and an angular coordinate $0 \leq \theta \leq 2\pi$ and recalling that the area element is $d\vec{A} = r\mathrm{d}r\mathrm{d}\theta\hat{\mathbf{z}}$. We obtain

$$\begin{aligned}\int_A \left(\vec{\nabla}\times\vec{G}\right)\cdot \mathrm{d}\vec{A} &= \int_0^{2\pi}\int_0^{1} 2\hat{\mathbf{z}}\cdot\hat{\mathbf{z}}r\mathrm{d}r\mathrm{d}\theta \\ &= 2\cdot\frac{1}{2}\cdot 2\pi \\ &= 2\pi, \end{aligned} \tag{26}$$

as before.

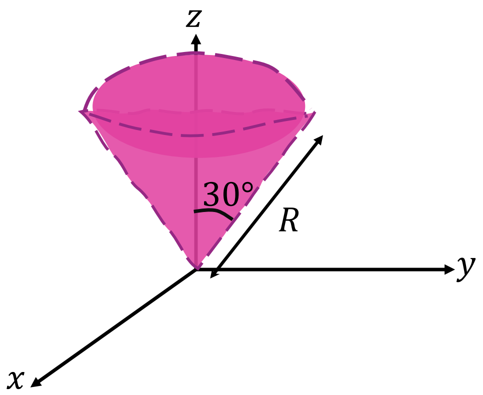

Fig. 2: The "ice-cream cone" surface S

Problem 3

Check Gauss's theorem

$$\int_S \vec{v}\cdot \mathrm{d}\vec{S} = \int_V \left(\vec{\nabla}\cdot\vec{v}\right)\mathrm{d}V, \tag{27}$$

where V is the volume inside a surface S, for $\vec{v} = r^2\sin\theta\hat{\mathbf{r}} + 4r^2\cos\theta\hat{\boldsymbol{\theta}} + r^2\tan\theta\hat{\boldsymbol{\varphi}}$ (in spherical coordinates (r,θ,φ)) and the surface S of the "ice cream cone" (figure 2).

Solution

The surface of the ice cream cone is composed of two parts,

(i) the dome at the top of the cone with $r = R$, $0 \le \theta \le \pi/6$,$0 \le \varphi \le 2\pi$,

(ii) body of the cone with $0 \le r \le R, \theta = \pi/6, 0 \le \varphi \le 2\pi$.

For the top surface where $r = R$ is constant, the area element will be $d\vec{A}_{\text{dome}} = R^2\sin\theta\mathrm{d}\theta\mathrm{d}\varphi\hat{\mathbf{r}}$. The body of the cone has a constant $\theta = \pi/6$, and therefore its area element will be $d\vec{A}_{\text{body}} = r\sin\pi/6\mathrm{d}r\mathrm{d}\varphi\hat{\boldsymbol{\theta}}$[1]. Hence, the scalar products between $\vec{v}$ and the area elements are

$$\vec{v}\,(r=R)\cdot \mathrm{d}\vec{A}_{\text{dome}} = R^4\sin^2\theta\mathrm{d}\theta\mathrm{d}\varphi, \tag{28}$$

$$\vec{v}\,(\theta=\pi/6)\cdot \mathrm{d}\vec{A}_{\text{body}} = \sqrt{3}r^3\mathrm{d}r\mathrm{d}\varphi. \tag{29}$$

[1] The area of a parallelogram with sides $\vec{a},\vec{b}$ is $\vec{a}\times\vec{b}$. Hence, the area element of a sphere with radius r is $d\vec{A} = (r\mathrm{d}\theta)\,\hat{\boldsymbol{\theta}}\times(r\sin\theta\mathrm{d}\varphi)\,\hat{\boldsymbol{\varphi}} = r^2\sin\theta\mathrm{d}\theta\mathrm{d}\varphi\hat{\mathbf{r}}$. Similarly, the area element of a cone is $d\vec{A} = (r\sin\theta\mathrm{d}\varphi)\,\hat{\boldsymbol{\varphi}}\times(\mathrm{d}r)\,\hat{\mathbf{r}} = r\sin\theta\mathrm{d}\varphi\mathrm{d}r\hat{\boldsymbol{\theta}}$.

We can now write the surface integral,

$$\int_S \vec{v}\cdot \mathrm{d}\vec{S} = \sqrt{3}\int_0^{2\pi} \mathrm{d}\varphi \int_0^R \mathrm{d}r r^3 + R^4 \int_0^{2\pi} \mathrm{d}\varphi \int_0^{\pi/6} \mathrm{d}\theta \sin^2\theta$$
$$= 2\pi\sqrt{3}\frac{R^4}{4} + 2\pi R^4 \int_0^{\pi/6} \mathrm{d}\theta \left(\frac{1-\cos(2\theta)}{2}\right)$$
$$= \pi R^4 \left(\frac{\sqrt{3}}{2} + \frac{\pi}{6} - \frac{1}{2}\sin\left(\frac{\pi}{3}\right)\right)$$
$$= \pi R^4 \left(\frac{\sqrt{3}}{4} + \frac{\pi}{6}\right). \tag{30}$$

We will now repeat the computation for the volume integral, expecting to find the same result.

The volume element in spherical coordinates is $\mathrm{d}V = r^2 \sin\theta \mathrm{d}r\mathrm{d}\theta\mathrm{d}\varphi$[2]. We begin by writing $\vec{\nabla}\cdot\vec{v}$ in spherical coordinates:

$$\vec{\nabla}\cdot\vec{v} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 v_r\right) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta v_\theta\right) + \frac{1}{r\sin\theta}\frac{\partial v_\varphi}{\partial\varphi}$$
$$= \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^4\sin\theta\right) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left(4r^2\sin\theta\cos\theta\right) + \frac{1}{r\sin\theta}\frac{\partial\left(r^2\tan\theta\right)}{\partial\varphi}$$
$$= 4r\sin\theta + \frac{4r}{\sin\theta}\left(\cos^2\theta - \sin^2\theta\right). \tag{31}$$

We can now complete the integration,

$$\int_V \left(\vec{\nabla}\cdot\vec{v}\right)\mathrm{d}V = \int_0^R \mathrm{d}r \int_0^{2\pi}\mathrm{d}\varphi \int_0^{\pi/6}\mathrm{d}\theta \left(4r\sin\theta + \frac{4r}{\sin\theta}\left(\cos^2\theta - \sin^2\theta\right)\right) r^2\sin\theta$$
$$= 4\int_0^R r^3\mathrm{d}r \int_0^{2\pi}\mathrm{d}\varphi \int_0^{\pi/6}\mathrm{d}\theta\cos^2\theta$$
$$= 4\cdot 2\pi\cdot\frac{R^4}{4}\int_0^{\pi/6}\mathrm{d}\theta\frac{1}{2}\left(1+\cos 2\theta\right)$$
$$= \pi R^4\left(\frac{\pi}{6} + \frac{\sqrt{3}}{4}\right). \tag{32}$$

Indeed, the integrals are equal!

[2] The volume of a parallelepiped with sides $\vec{a}, \vec{b}, \vec{c}$ is $\vec{c}\cdot\left(\vec{a}\times\vec{b}\right)$. Hence, the volume element of a ball with radius r is $\mathrm{d}r\hat{\mathbf{r}}\cdot\mathrm{d}\vec{A}_{\text{sphere}} = r^2\sin\theta\mathrm{d}\theta\mathrm{d}\varphi\mathrm{d}r$.

Problem 4

Prove that for any vector $\vec{A}$ and scalar function f,

1. $\vec{\nabla} \cdot \left(\vec{\nabla} \times \vec{A}\right) = 0,$
2. $\vec{\nabla} \times \left(\vec{\nabla} f\right) = 0.$

Solution

We will use Cartesian coordinates to prove this relation, eventually concluding that it holds in general (since the expression is independent of coordinates). We will give two solutions, first explicitly and later in summation notation.

1. We begin with explicitly writing the vectors involved:

$$\begin{aligned}\vec{\nabla} \cdot \left(\vec{\nabla} \times \vec{A}\right) &= \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right) \cdot \left(\frac{\partial}{\partial y} V_z - \frac{\partial}{\partial z} V_y, \frac{\partial}{\partial z} V_x - \frac{\partial}{\partial x} V_z, \frac{\partial}{\partial x} V_y - \frac{\partial}{\partial y} V_x\right) \\ &= \frac{\partial}{\partial x}\left(\frac{\partial}{\partial y} V_z - \frac{\partial}{\partial z} V_y\right) + \frac{\partial}{\partial y}\left(\frac{\partial}{\partial z} V_x - \frac{\partial}{\partial x} V_z\right) + \frac{\partial}{\partial z}\left(\frac{\partial}{\partial x} V_y - \frac{\partial}{\partial y} V_x\right) \\ &= \frac{\partial}{\partial x}\frac{\partial}{\partial y} V_z - \frac{\partial}{\partial x}\frac{\partial}{\partial z} V_y + \frac{\partial}{\partial y}\frac{\partial}{\partial z} V_x - \frac{\partial}{\partial y}\frac{\partial}{\partial x} V_z + \frac{\partial}{\partial z}\frac{\partial}{\partial x} V_y - \frac{\partial}{\partial z}\frac{\partial}{\partial y} V_x.\end{aligned} \tag{33}$$

We gather terms,

$$\vec{\nabla} \cdot \left(\vec{\nabla} \times \vec{A}\right) = \left(\frac{\partial}{\partial y}\frac{\partial}{\partial z} - \frac{\partial}{\partial z}\frac{\partial}{\partial y}\right) V_x + \left(\frac{\partial}{\partial z}\frac{\partial}{\partial x} - \frac{\partial}{\partial x}\frac{\partial}{\partial z}\right) V_y + \left(\frac{\partial}{\partial x}\frac{\partial}{\partial y} - \frac{\partial}{\partial y}\frac{\partial}{\partial x}\right) V_z. \tag{34}$$

Lastly, we recall that Cartesian derivatives have exchange symmetry, and therefore all bracketed terms vanish. Hence, $\vec{\nabla} \cdot \left(\vec{\nabla} \times \vec{A}\right) = 0$.
In summation notation,

$$\vec{\nabla} \cdot \left(\vec{\nabla} \times \vec{A}\right) = \frac{\partial}{\partial x_i}\left(\vec{\nabla} \times \vec{A}\right)_i = \frac{\partial}{\partial x_i}\left(\varepsilon_{ijk}\frac{\partial}{\partial x_j} A_k\right). \tag{35}$$

Since the derivative only operates on $\vec{A}$,

$$\frac{\partial}{\partial x_i}\left(\varepsilon_{ijk}\frac{\partial}{\partial x_j} A_k\right) = \varepsilon_{ijk}\frac{\partial}{\partial x_i}\frac{\partial}{\partial x_j} A_k. \tag{36}$$

We use the cyclic property of the Levi-Civita symbol and the exchange symmetry of the Cartesian derivatives to obtain

$$\varepsilon_{ijk}\frac{\partial}{\partial x_i}\frac{\partial}{\partial x_j}A_k = \varepsilon_{jki}\frac{\partial}{\partial x_i}\frac{\partial}{\partial x_j}A_k = \varepsilon_{jki}\frac{\partial}{\partial x_j}\frac{\partial}{\partial x_i}A_k. \quad (37)$$

The expression $\varepsilon_{jki}\left(\partial/\partial x_i\right)A_k$ is almost $\left(\vec{\nabla}\times\vec{A}\right)_j = \varepsilon_{jik}\left(\partial/\partial x_i\right)A_k$, which we can fix by using the anti-cyclic property of ε_{ijk} again:

$$\varepsilon_{jki}\frac{\partial}{\partial x_j}\frac{\partial}{\partial x_i}A_k = -\varepsilon_{jik}\frac{\partial}{\partial x_j}\frac{\partial}{\partial x_i}A_k = -\frac{\partial}{\partial x_j}\left(\vec{\nabla}\times\vec{A}\right)_j. \quad (38)$$

Since the only object that is equal to its negative counterpart is zero, we conclude that

$$\vec{\nabla}\cdot\left(\vec{\nabla}\times\vec{A}\right) = 0. \quad (39)$$

2. We write the explicit expression,

$$\vec{\nabla}\times\left(\vec{\nabla}f\right) = \left(\frac{\partial}{\partial x},\frac{\partial}{\partial y},\frac{\partial}{\partial z}\right)\times\left(\frac{\partial f}{\partial x},\frac{\partial f}{\partial y},\frac{\partial f}{\partial z}\right), \quad (40)$$

and use the determinant rule to compute the curl:

$$\vec{\nabla}\times\vec{V} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \frac{\partial}{\partial x_i} & \frac{\partial}{\partial x_j} & \frac{\partial}{\partial x_k} \\ V_i & V_j & V_k \end{vmatrix} = \left(\frac{\partial}{\partial x_j}V_k - \frac{\partial}{\partial x_k}V_j\right)\hat{\mathbf{i}} + \left(\frac{\partial}{\partial x_i}V_k - \frac{\partial}{\partial x_k}V_i\right)\hat{\mathbf{j}} + \left(\frac{\partial}{\partial x_i}V_j - \frac{\partial}{\partial x_j}V_i\right)\hat{\mathbf{k}}. \quad (41)$$

Applying this rule to our expression, we find

$$\vec{\nabla}\times\left(\vec{\nabla}f\right) = \left(\frac{\partial}{\partial y}\frac{\partial f}{\partial z} - \frac{\partial}{\partial z}\frac{\partial f}{\partial y}\right)\hat{\mathbf{x}} + \left(\frac{\partial}{\partial x}\frac{\partial f}{\partial z} - \frac{\partial}{\partial z}\frac{\partial f}{\partial x}\right)\hat{\mathbf{y}} + \left(\frac{\partial}{\partial x}\frac{\partial f}{\partial y} - \frac{\partial}{\partial y}\frac{\partial f}{\partial x}\right)\hat{\mathbf{z}}. \quad (42)$$

The exchange symmetry of the Cartesian derivatives cancels each term in the brackets, and we get $\vec{\nabla}\times\left(\vec{\nabla}f\right) = 0$. In summation notation,

$$\left(\vec{\nabla}\times\left(\vec{\nabla}f\right)\right)_i = \varepsilon_{ijk}\frac{\partial}{\partial x_j}\left(\vec{\nabla}f\right)_k = \varepsilon_{ijk}\frac{\partial}{\partial x_j}\left(\frac{\partial}{\partial x_k}f\right). \quad (43)$$

We again use the exchange symmetry of the derivatives to obtain

$$\varepsilon_{ijk}\frac{\partial}{\partial x_j}\left(\frac{\partial}{\partial x_k}f\right) = \varepsilon_{ijk}\frac{\partial}{\partial x_k}\frac{\partial}{\partial x_j}f, \quad (44)$$

and apply the anti-cyclic property of the Levi-Civita symbol,

$$\varepsilon_{ijk}\frac{\partial}{\partial x_k}\frac{\partial}{\partial x_j}f = -\varepsilon_{ikj}\frac{\partial}{\partial x_k}\frac{\partial}{\partial x_j}f. \tag{45}$$

Converting back to vector notation, we have that

$$\left(\vec{\nabla} \times \left(\vec{\nabla} f\right)\right)_i = -\left(\vec{\nabla} \times \left(\vec{\nabla} f\right)\right)_i, \tag{46}$$

and therefore $\left(\vec{\nabla} \times \left(\vec{\nabla} f\right)\right) = 0$.