Homework 2

Question 1

The "step" function, also known as Heaviside function, is defined as

$$\Theta(x) = \begin{cases} 1, & x > 0, \\ 0, & x < 0. \end{cases}$$

Show that

$$\Theta'(x) = \delta(x).$$

Question 2

Prove Green's first and second identities,

$$\iiint_V \left(\psi \nabla^2 f + \vec{\nabla} f \vec{\nabla} \psi\right) \mathrm{d}V = \oiint_{\partial V} \psi \frac{\partial f}{\partial n} \mathrm{d}S,$$

$$\iiint_V \left(\psi \nabla^2 f - f \nabla^2 \psi\right) \mathrm{d}V = \oiint_{\partial V} \left[\psi \frac{\partial f}{\partial n} - f \frac{\partial \psi}{\partial n}\right] \mathrm{d}S,$$

where $\partial\psi/\partial n \equiv \vec{\nabla}\psi \cdot \hat{\mathbf{n}}$ and $\hat{\mathbf{n}}$ is an outward pointing unit normal on the surface S.

Remark Green's theorems are used to prove uniqueness of the boundary value problem for the Laplace operator, subject to Dirichlet or Neumann boundary conditions.

Question 3

Given the functions $f(x) = x^2 - 3x - 4$, $g(x) = \arcsin(x)$, find

$$\int_{-\infty}^{0} g(x)\,\delta(f(x))\,dx.$$

Question 4

A point charge q is placed at a distance $d > R$ outside a conducting sphere with radius R. Find the charge density σ on the sphere.

Question 5

Consider a uniformly charged ellipsoid, with volume

$$V_{\text{ell}} = \left\{ x, y, z : \frac{x^2}{a^2} + \frac{y^2}{a^2} + \frac{z^2}{b^2} \leq 1 \right\},$$

and total charge Q. Recall the Quadrupole tensor,

$$D_{ij} = \int \left(3x_i' x_j' - r'^2 \delta_{ij}\right) \rho(r')\, \mathrm{d}^3 r'.$$

Find the Quadrupole moment (with respect to the origin) and the potential of the ellipsoid.