# Homework 2

## Question 1

The "step" function, also known as Heaviside function, is defined as

$$\Theta(x) = \begin{cases} 1, & x > 0, \\ 0, & x < 0. \end{cases}$$

Show that

$$\Theta'(x) = \delta(x).$$

## Answer

Notice that for $x \neq 0$, $\Theta'(x) = 0$. Intuitively, we know that the derivative of a step function diverges at the point of the step. However, we still need to show that $\Theta'(x)$ satisfies the integral property of the delta function.

We look at an integral of the form

$$\int_{-\infty}^{\infty} \Theta'(x) f(x) \, \mathrm{d}x.$$

We can evaluate such an integral using integration by parts,

$$\begin{aligned} \int_{-\infty}^{\infty} \Theta'(x) f(x) \, \mathrm{d}x &= \Theta(x) f(x)\big|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} \Theta(x) f'(x) \, \mathrm{d}x \\ &= f(\infty) - \int_{0}^{\infty} f'(x) \, \mathrm{d}x \\ &= f(\infty) - (f(\infty) - f(0)) \\ &= f(0). \end{aligned}$$

We find that $\Theta'(x)$ acts the same as a delta function under integration, and therefore,

$$\Theta'(x) = \delta(x).$$

Notice also that

$$\int_{-\infty}^{x} \delta(s)\,\mathrm{d}s = \begin{cases} 0, & x < 0, \\ 1, & x > 0, \end{cases} = \Theta(x).$$

## Question 2

Prove Green's first and second identities,

$$\iiint_V \left(\psi \nabla^2 f + \vec{\nabla} f \vec{\nabla} \psi\right) \mathrm{d}V = \oiint_{\partial V} \psi \frac{\partial f}{\partial n} \mathrm{d}S,$$

$$\iiint_V \left(\psi \nabla^2 f - f \nabla^2 \psi\right) \mathrm{d}V = \oiint_{\partial V} \left[\psi \frac{\partial f}{\partial n} - f \frac{\partial \psi}{\partial n}\right] \mathrm{d}S,$$

where $\partial\psi/\partial n \equiv \vec{\nabla}\psi \cdot \hat{\mathbf{n}}$ and $\hat{\mathbf{n}}$ is an outward pointing unit normal on the surface $S$.

**Remark** Green's theorems are used to prove uniqueness of the boundary value problem for the Laplace operator, subject to Dirichlet or Neumann boundary conditions.

## Solution

$$\begin{aligned} \iiint_V \left(\psi \nabla^2 f + \vec{\nabla} f \vec{\nabla} \psi\right) \mathrm{d}V &= \iiint_V \left(\vec{\nabla} \cdot \left(\psi \vec{\nabla} f\right) - \cancel{\vec{\nabla}\psi \vec{\nabla} f + \vec{\nabla} f \vec{\nabla} \psi}\right) \mathrm{d}V \\ &= \oiint_{\partial V} \psi \vec{\nabla} f \mathrm{d}\vec{S} \\ &= \oiint_{\partial V} \psi \vec{\nabla} f \hat{\mathbf{n}} \mathrm{d}S = \oiint_{\partial V} \psi \frac{\partial f}{\partial n} \mathrm{d}S, \end{aligned}$$

$$\iiint_V \left(\psi\nabla^2 f - f\nabla^2\psi\right) \mathrm{d}V = \iiint_V \left(\vec{\nabla}\cdot\left(\psi\vec{\nabla} f\right) - \vec{\nabla}\psi\vec{\nabla} f - \left(\vec{\nabla}\cdot\left(f\vec{\nabla}\psi\right) - \vec{\nabla} f\vec{\nabla}\psi\right)\right) \mathrm{d}V$$
$$= \iiint_V \vec{\nabla}\cdot\left(\psi\vec{\nabla} f - f\vec{\nabla}\psi\right) \mathrm{d}V$$
$$= \oiint_{\partial V} \left(\psi\vec{\nabla} f - f\vec{\nabla}\psi\right)\cdot \mathrm{d}\vec{S}$$
$$= \oiint_{\partial V} \left(\psi\vec{\nabla} f - f\vec{\nabla}\psi\right)\cdot \hat{\mathbf{n}}\mathrm{d}S$$
$$= \oiint_{\partial V} \left(\psi\frac{\partial f}{\partial n} - f\frac{\partial \psi}{\partial n}\right) \mathrm{d}S.$$

# Question 3

Given the functions $f(x) = x^2 - 3x - 4$, $g(x) = \arcsin(x)$, find

$$\int_{-\infty}^{0} g(x)\,\delta(f(x))\,dx.$$

# Solution

The function $f$ has the following roots:

$$x_{1,2} = 4, -1.$$

We notice that the positive root is not within the bounds of integration $\infty \leq x \leq 0$. We use the identity

$$\delta(f(x)) = \sum_{i=1}^{2} \frac{\delta(x - x_i)}{|f'(x_i)|} = \frac{\delta(x-4)}{5} + \frac{\delta(x+1)}{5},$$

and get

$$\int_{-\infty}^{0} g(x)\,\delta(f(x))\,dx = \frac{1}{5}\int_{-\infty}^{0} \arcsin(x)\,\delta(x+1)\,dx$$
$$= \frac{1}{5}\arcsin(-1) = -\frac{\pi}{10}.$$

# Question 4

A point charge $q$ is placed at a distance $d > R$ outside a conducting sphere with radius $R$. Find the charge density $\sigma$ on the sphere.

## Solution

To compute the charge density on the sphere, we are going to find the potential outside the sphere by using an image charge. Then, we will find $E_n = \hat{\mathbf{n}} \cdot \vec{E}\,(r = R)$ on the sphere, and use $E_n = 2\pi\sigma$.

The sphere is an equipotential surface, and we choose that constant potential to be zero. To obtain this boundary condition, from symmetry, another charge $q'$ must be place inside the sphere (say, at a distance $d'$ from the origin), somewhere on the ray which connects the origin and the charge $q$. Then, the potential of the setup outside the sphere is

$$\Phi\left(\vec{r}\right) = \frac{q}{4\pi\left|\vec{r} - \vec{d}\right|} + \frac{q'}{4\pi\left|\vec{r} - \vec{d'}\right|}.$$

Since the potential vanishes at $|r| = R$, we have

$$\begin{aligned} 0 &= \frac{q}{4\pi\left|R\hat{\mathbf{r}} - \vec{d}\right|} + \frac{q'}{4\pi\left|R\hat{\mathbf{r}} - \vec{d'}\right|} \\ &= \frac{q}{4\pi\left|R\hat{\mathbf{r}} - d\hat{\mathbf{d}}\right|} + \frac{q'}{4\pi\left|R\hat{\mathbf{r}} - d'\hat{\mathbf{d}}\right|} \\ &= \frac{q}{4\pi R\left|\hat{\mathbf{r}} - \frac{d}{R}\hat{\mathbf{d}}\right|} + \frac{q'}{4\pi d'\left|\frac{R}{d'}\hat{\mathbf{r}} - \hat{\mathbf{d}}\right|} \\ &= \frac{q}{4\pi R\sqrt{1 - \frac{d}{R}\hat{\mathbf{r}} \cdot \hat{\mathbf{d}} + \left(\frac{d}{R}\right)^2}} + \frac{q'}{4\pi d'\sqrt{1 - \frac{R}{d'}\hat{\mathbf{r}} \cdot \hat{\mathbf{d}} + \left(\frac{R}{d'}\right)^2}}. \end{aligned}$$

From this form we can infer that the distance and charge which satisfy $\Phi\left(r = R\right) = 0$ are

$$d' = \frac{R^2}{d}, q' = -q\frac{d'}{R} = -q\frac{R}{d}.$$

We therefore have the potential

$$\Phi\left(\vec{r}\right) = \frac{q}{4\pi}\left[\frac{1}{\left|r\hat{\mathbf{r}} - d\hat{\mathbf{d}}\right|} - \frac{R/d}{\left|r\hat{\mathbf{r}} - \frac{R^2}{d}\hat{\mathbf{d}}\right|}\right],$$

and thus the field on the sphere, in the $\hat{\mathbf{r}}$ direction (the normal to the sphere) is

$$\begin{aligned}
\vec{E}\cdot\hat{\mathbf{r}} &= -\left.\frac{\partial\Phi}{\partial r}\right|_{r=R} \\
&= -\frac{q}{8\pi}\left[\frac{2R-2d\hat{\mathbf{r}}\cdot\hat{\mathbf{d}}}{\left(R^2+d^2-2Rd\hat{\mathbf{r}}\cdot\hat{\mathbf{d}}\right)^{3/2}}-\frac{R}{d}\frac{2R-2\frac{R^2}{d}\hat{\mathbf{r}}\cdot\hat{\mathbf{d}}}{\left(R^2+\left(\frac{R^2}{d}\right)^2-2R\frac{R^2}{d}\hat{\mathbf{r}}\cdot\hat{\mathbf{d}}\right)^{3/2}}\right] \\
&= -\frac{q}{8\pi}\left[\frac{2R-2d\cos\alpha}{d^3\left(1+\left(\frac{R}{d}\right)^2-2\frac{R}{d}\cos\alpha\right)^{3/2}}-\frac{R}{d}\frac{2R-2\frac{R^2}{d}\cos\alpha}{R^3\left(1+\left(\frac{R}{d}\right)^2-2\frac{R}{d}\cos\alpha\right)^{3/2}}\right] \\
&= -\frac{q}{4\pi d^2}\frac{R}{d}\frac{1-\left(\frac{d}{R}\right)^2}{\left(1+\left(\frac{R}{d}\right)^2-\frac{R}{d}\cos\alpha\right)^{3/2}}.
\end{aligned}$$

Dividing this result by $2\pi$ results in $\sigma$.

# Question 5

Consider a uniformly charged ellipsoid, with volume

$$V_{\text{ell}} = \left\{x,y,z:\frac{x^2}{a^2}+\frac{y^2}{a^2}+\frac{z^2}{b^2}\leq 1\right\},$$

and total charge $Q$. Recall the Quadrupole tensor,

$$D_{ij} = \int\left(3x'_i x'_j - r'^2\delta_{ij}\right)\rho\left(r'\right)\mathrm{d}^3r'.$$

Find the Quadrupole moment (with respect to the origin) and the potential of the ellipsoid.

# Solution

Note that

$$\rho = \frac{Q}{4\pi a^2 b/3}.$$

Consider

$$D_{xy} = \int 3xy\rho\mathrm{d}x\mathrm{d}y\mathrm{d}z.$$

Replacing $\widetilde{x}=x/a, \widetilde{y}=y/a, \widetilde{z}=z/b$, we obtain

$$\begin{aligned}
D_{xy} &= \frac{9a^2Q}{4\pi}\int xy\mathrm{d}\widetilde{x}\mathrm{d}\widetilde{y}\mathrm{d}\widetilde{z} \\
&= \frac{9a^2Q}{4\pi}\int \widetilde{r}^2\mathrm{d}\widetilde{r}\int \sin^2\theta\mathrm{d}\theta\overbrace{\int_0^{2\pi}\cos\varphi\sin\varphi\mathrm{d}\varphi}^{=0} \\
&= 0.
\end{aligned}$$

This will be the same result for $D_{yz}, D_{xz}$. Now, from the symmetry of the Ellipsoid in the $x-y$ plane, $D_{xx}=D_{yy}$. In class, you proved that $\mathrm{Tr}D=0$,which means we also have

$$D_{zz}=-2D_{xx}.$$

Therefore, we need only compute $D_{xx}$:

$$\begin{aligned}
D_{xx} &= \int\left(3x^2-r^2\right)\rho\mathrm{d}^3r \\
&= \int\left(2x^2-y^2-z^2\right)\rho\mathrm{d}^3r \\
&= \frac{3Q}{4\pi a^2b}\int\left(2a^2\widetilde{x}^2-a^2\widetilde{y}^2-b^2\widetilde{z}^2\right)\mathrm{d}^3\widetilde{r} \\
&= \frac{3Q}{4\pi a^2b}\int_0^1\int_{\theta=0}^{\pi}\int_{\varphi=0}^{2\pi}\widetilde{r}^2\sin\theta\mathrm{d}\widetilde{r}\mathrm{d}\theta\mathrm{d}\varphi \\
&\times\left(2a^2\widetilde{r}^2\sin^2\theta\cos^2\varphi-a^2\widetilde{r}^2\sin^2\theta\sin^2\varphi-b^2\widetilde{r}^2\cos^2\theta\right) \\
&= \frac{Q}{10a^2b}\left(2a^2-b^2\right).
\end{aligned}$$

We find the potential through the multipole expansion, so we need the dipole moments:

$$\begin{aligned}
P_x &= \int x\rho\mathrm{d}x\mathrm{d}y\mathrm{d}z \\
&= \rho a^3b\int\widetilde{x}\mathrm{d}\widetilde{x}\mathrm{d}\widetilde{y}\mathrm{d}\widetilde{z} \\
&= \rho a^3b\int\widetilde{r}^3\mathrm{d}\widetilde{r}\int\sin^2\theta\mathrm{d}\theta\overbrace{\int\cos\varphi\mathrm{d}\varphi}^{=0}=0.
\end{aligned}$$

The same for $P_y$ from the symmetry of the problem.

$$
\begin{aligned}
P_z &= \int z\rho \mathrm{d}x\mathrm{d}y\mathrm{d}z \\
&= a^2 b^2 \rho \int \widetilde{r}^3 \mathrm{d}\widetilde{r} \overbrace{\int_0^{2\pi} \sin\theta \mathrm{d}\theta}^{=0} \int \mathrm{d}\varphi = 0.
\end{aligned}
$$

Therefore, for a point far away where $x, y, z \gg a, b$:

$$
\Phi\left(x, y, z\right) \approx \frac{Q}{\sqrt{x^2+y^2+z^2}} + \frac{Q\left(2a^2-b^2\right)}{20a^2 b\left(x^2+y^2+z^2\right)^{5/2}}\left(x^2+y^2-2z^2\right).
$$