Class Exercise 3 - Green function, separation of variables and Multipole expansion

The Green function

In many cases we wish to solve differential equations of the form

$$\hat{L}f\left(\vec{r}\right)=S\left(\vec{r}\right), \tag{1}$$

where $\hat{L}$ is a linear differential operator and $S\left(\vec{r}\right)$ is a source function. A prominent example in this course is the Poisson equation,

$$\overbrace{\nabla^2}^{\text{linear op.}}\Phi=\overbrace{-4\pi\rho}^{\text{source}}.$$

The method of solving such equations relies on the fact that, much like matrices in linear algebra (which act on discrete indices $M_{ij}V_j=U_i$) , a linear differential operator (which acts on continuous variables) is invertible, in the sense that there exists an operator $\hat{L}^{-1}$, such that

$$\hat{L}^{-1}\hat{L}=1. \tag{2}$$

We call $\hat{L}^{-1}$ the Green function $G\left(\vec{r},\vec{r}'\right)$, and it solves the continuous version of equation (2),

$$\hat{L}G\left(\vec{r},\vec{r}'\right)=4\pi\delta\left(\vec{r}-\vec{r}'\right).$$

The solution to equation (1) will therefore be

$$f\left(\vec{r}\right)=\hat{L}^{-1}S\left(\vec{r}\right)=\int \mathrm{d}\vec{r}'G\left(\vec{r},\vec{r}'\right)S\left(\vec{r}'\right).$$

In class, the general Green function of the Laplace operator ∇^2 was found:

$$G\left(\vec{r},\vec{r}'\right)=G\left(\vec{r}-\vec{r}'\right)=\frac{1}{\left|\vec{r}-\vec{r}'\right|}+F\left(\vec{r},\vec{r}'\right),$$

where $F(\vec{r},\vec{r}')$ is some function satisfying the Laplace equation in the volume V,

$$\nabla^2 F(\vec{r},\vec{r}') = 0.$$

Problem 1

Use the Poisson equation

$$\nabla^2\Phi = -4\pi\rho,$$

and the second Green identity (HW2)

$$\iiint_V \left(\psi\nabla^2 f - f\nabla^2\psi\right)\mathrm{d}V = \oiint_{\partial V}\left[\psi\frac{\partial f}{\partial n} - f\frac{\partial\psi}{\partial n}\right]\mathrm{d}S,$$

to determine that the solution to the potential is

$$\Phi(\vec{r}) = \iiint_V \rho(\vec{r}')\,G(\vec{r},\vec{r}')\,\mathrm{d}V' + \frac{1}{4\pi}\oiint_{\partial V}\left(G(\vec{r},\vec{r}')\frac{\partial\Phi}{\partial n'} - \Phi\frac{\partial G(\vec{r},\vec{r}')}{\partial n'}\right)\mathrm{d}S'.$$

Solution

We apply the identity to $\psi = \Phi$ and $f = G(\vec{r},\vec{r}')$,

$$\iiint_V \left(\Phi(\vec{r}')\,\nabla^2 G(\vec{r},\vec{r}') - G(\vec{r},\vec{r}')\,\nabla^2\Phi(\vec{r}')\right)\mathrm{d}V' = \oiint_{\partial V}\left[\Phi(\vec{r}')\,\frac{\partial G(\vec{r},\vec{r}')}{\partial n} - G(\vec{r},\vec{r}')\,\frac{\partial\Phi(\vec{r}')}{\partial n}\right]\mathrm{d}S'.$$

In the first term on the LHS, we can use $\nabla^2 G(\vec{r},\vec{r}') = -4\pi\delta^{(3)}(\vec{r}-\vec{r}')$ to obtain

$$\iiint_V \Phi(\vec{r}')\,\nabla^2 G(\vec{r},\vec{r}')\,\mathrm{d}V' = -4\pi\iiint_V \Phi(\vec{r}')\,\delta^{(3)}(\vec{r}-\vec{r}')\,\mathrm{d}V' = -4\pi\Phi(\vec{r}),$$

and in the second term we insert the Poisson equation,

$$-\iiint_V G(\vec{r},\vec{r}')\,\nabla^2\Phi(\vec{r}')\,\mathrm{d}V' = 4\pi\iiint_V G(\vec{r},\vec{r}')\,\rho(\vec{r}')\,\mathrm{d}V'.$$

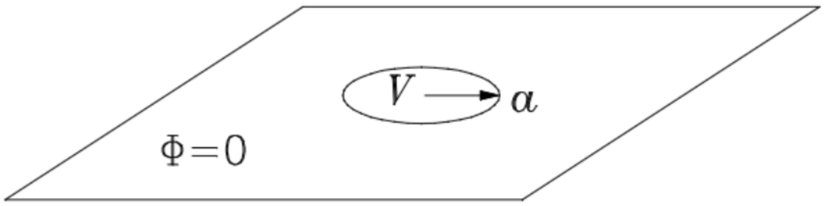
Rearranging terms, we obtain the general solution to the Poisson equation in the desired form,

$$\Phi(\vec{r}) = \iiint_V \rho(\vec{r}')\,G(\vec{r},\vec{r}')\,\mathrm{d}V' + \frac{1}{4\pi}\oiint_{\partial V}\left(G(\vec{r},\vec{r}')\,\frac{\partial\Phi(\vec{r}')}{\partial n'} - \Phi(\vec{r}')\,\frac{\partial G(\vec{r},\vec{r}')}{\partial n'}\right)\mathrm{d}S'.$$

Problem 2

Consider an infinite plate in the $x - y$ plane, held at fixed potential (see figure)

$$\Phi(r) = \begin{cases} V, & r < a, \\ 0, & r > a. \end{cases}$$



Find the potential at some point $z > 0$ on the symmetry axis.

Solution

We use the solution from the last section, where we proved that

$$\Phi(\vec{r}) = \iiint_V \rho(\vec{r}')\, G(\vec{r}, \vec{r}')\, \mathrm{d}V' + \frac{1}{4\pi} \oiint_{\partial V} \left(G(\vec{r}, \vec{r}') \frac{\partial \Phi(\vec{r}')}{\partial n'} - \Phi(\vec{r}') \frac{\partial G(\vec{r}, \vec{r}')}{\partial n'} \right) \mathrm{d}S'.$$

The unit normal to the surface is

$$n' = -z'.$$

Here, the value of the potential on the boundary (the plane) is given, and thus we have a Dirichlet boundary condition. In class you have seen that a Dirichlet boundary condition corresponds to the choice $G(z = 0) = 0$. Since we do not have any charges in the problem, $\rho = 0$ and we need only find

$$\Phi(\vec{r}) = -\frac{1}{4\pi} \oiint_{\partial V} \Phi(\vec{r}') \frac{\partial G(\vec{r}, \vec{r}')}{\partial n'} \mathrm{d}S',$$

where the Green's function solves

$$\begin{aligned} \nabla'^2 G(\vec{r}, \vec{r}') &= -4\pi\delta(\vec{r} - \vec{r}'), \\ G(\vec{r}, \vec{r}' \in \partial V) &= 0. \end{aligned}$$

We use the method of image charges to satisfy the boundary conditions on the plane. Note that the Green function above has the form of a single point charge above a grounded plane. In class you saw that the solution requires an identical image charge to be placed symmetrically below the plane. Therefore, due to uniqueness we may use the solution to that problem to solve ours!

We therefore consider two identical point charges with $q = 1$, located above and below the plane at symmetric distances $\pm z'$,

$$G(\vec{r}, \vec{r}') = \frac{1}{\sqrt{(x-x')^2 + (y-y')^2 + (z-z')^2}} - \frac{1}{\sqrt{(x-x')^2 + (y-y')^2 + (z+z')^2}}.$$

The derivative in the direction normal to the plane is then

$$\frac{\partial G}{\partial n'} = -\frac{\partial G}{\partial z'} = -\frac{z-z'}{\left((x-x')^2 + (y-y')^2 + (z-z')^2\right)^{3/2}} - \frac{z+z'}{\left((x-x')^2 + (y-y')^2 + (z+z')^2\right)^{3/2}},$$

and on the plane $(z' = 0)$ we have

$$\left.\frac{\partial G}{\partial n'}\right|_{z'=0} = -\frac{2z}{\left((x-x')^2 + (y-y')^2 + z^2\right)^{3/2}}.$$

We substitute the expression we found into the integral form of the potential and obtain

$$\begin{aligned}
\Phi(x=0, y=0, z>0) &= \frac{z}{2\pi} \int_{-\infty}^{\infty} \mathrm{d}x' \int_{-\infty}^{\infty} \mathrm{d}y' \frac{\Phi(x', y')}{\left((x-x')^2 + (y-y')^2 + z^2\right)^{3/2}} \\
&= \frac{z}{2\pi} \int_0^{2\pi} d\theta' \left[\int_0^{r'=a} \frac{V}{(r'^2 + z^2)^{3/2}} r' \mathrm{d}r' + \int_{r'=a}^{\infty} \frac{0}{(r'^2 + z^2)^{3/2}} r' \mathrm{d}r' \right] \\
&= V\left(1 - \frac{z}{\sqrt{z^2 + a^2}}\right).
\end{aligned}$$

Problem 3

Find the potential in the region $x \in [0, a]$ and $y > 0$ of the 2d well in the figure,

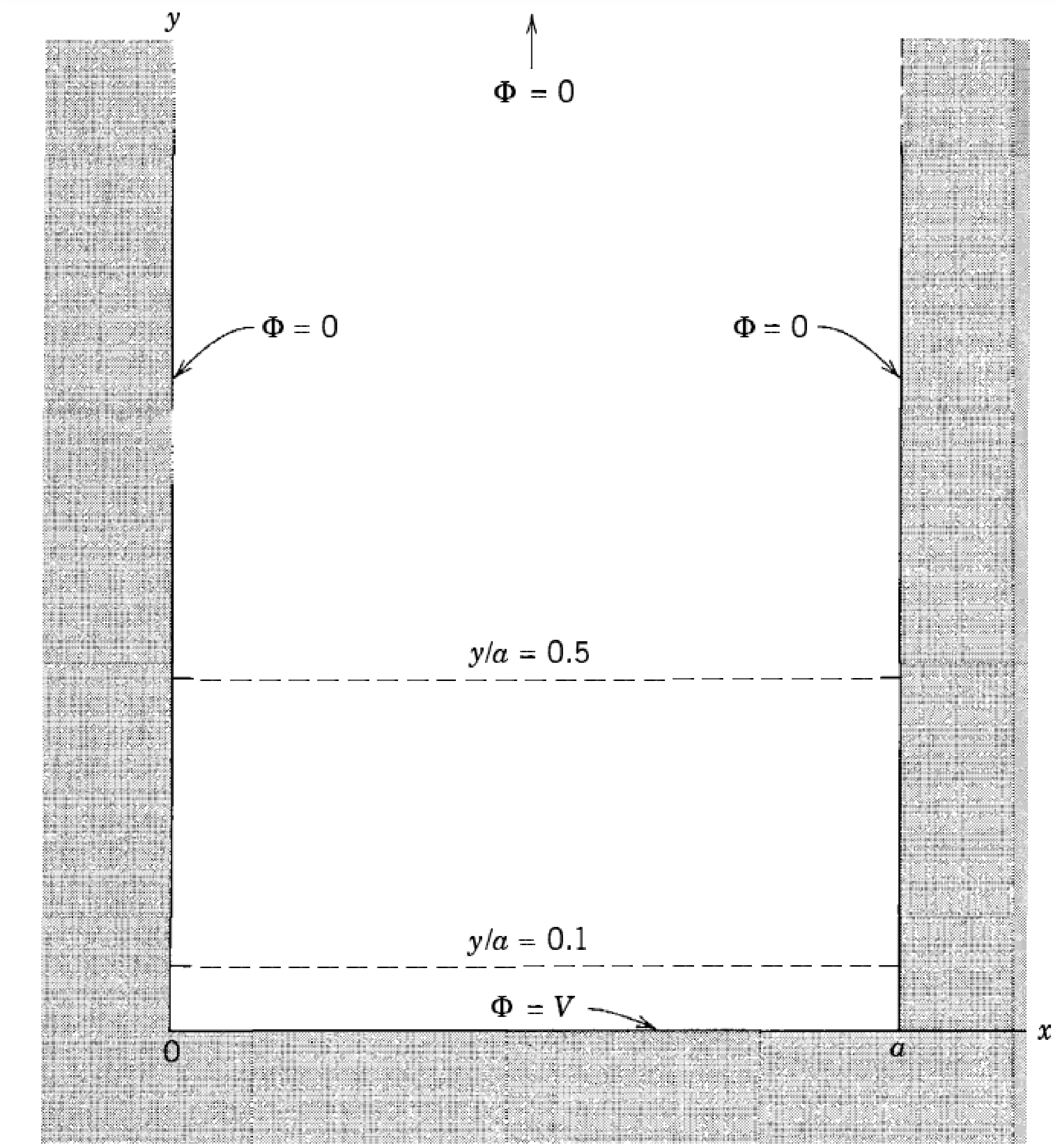


Fig. 1: Semi-infinite space between two parallel plates.

with the boundary conditions

$$\Phi(x, y = 0) = V, \qquad \Phi(x = 0, y) = \Phi(x = a, y) = 0.$$

Solution

We start with the Laplace equation,

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)\Phi\left(x,y\right)=0,$$

and use separation of variables with $\Phi\left(x,y\right)=X\left(x\right)Y\left(y\right)$ to obtain

$$\frac{1}{X}\frac{d^2X}{dx^2}+\frac{1}{Y}\frac{d^2Y}{dy^2}=0.$$

Since X,Y are functions of different variables, the only possible solution is

$$\frac{1}{X}\frac{d^2X}{dx^2}=C_x,\frac{1}{Y}\frac{d^2Y}{dy^2}=C_y,$$

where C_x,C_y are constants which satisfy $C_x+C_y=0$. We therefore have that $C_x=-C_y$. The equations we have obtained are *ordinary* differential equations (as opposed to the previous *partial* diff. equations), with solutions

$$\begin{cases} C\sin\left(cz\right)+D\cos\left(cz\right), & \text{if } c<0,\\ Ae^{cz}+Be^{-cz}, & \text{if } c>0.\end{cases}$$

Since the problem is periodic in x we expect $X\left(x\right)$ to be periodic, which would require $C_x<0$. Since $\Phi\left(x,y\to\infty\right)$ should vanish from standard physical considerations, we similarly expect $Y\left(y\right)$ to have an exponential form (in order to decay far away from $y=0$), and therefore write

$$\frac{1}{X}\frac{d^2X}{dx^2}=-\alpha^2=-\frac{1}{Y}\frac{d^2Y}{dy^2},$$

for $\alpha\in\mathbb{R}$. Hence, the solutions are

$$\begin{aligned} X\left(x\right)&=A\sin\left(\alpha x\right)+B\cos\left(\alpha x\right),\\ Y\left(y\right)&=C\exp\left(\alpha y\right)+D\exp\left(-\alpha y\right).\end{aligned}$$

To have $\Phi\left(x,y\to\infty\right)$ we require $C=0$ (for the choice $\alpha>0$). We apply the boundary conditions and obtain

$$\begin{aligned} &Y\left(y=0\right)=V=D,\\ &X\left(x=0\right)=0=B,\\ &X\left(x=a\right)=0=A\sin\left(\alpha a\right)\implies\alpha_n=\frac{\pi n}{a},n\in\mathbb{N}/\left\{0\right\}.\end{aligned}$$

Note that although $\sin(x) = 0$ for any $x = \pi b$, $b \in \mathbb{Z}$, we have specified above that $\alpha > 0$ and thus $n > 0$ as well. So far, we have the solution

$$\Phi(x, y) = \sum_{n=1}^{\infty} C_n \exp\left(-\frac{\pi n}{a} y\right) \sin\left(\frac{\pi n}{a} x\right),$$

and we just need to obtain the coefficients C_n. To do so, we use orthogonality of the sine basis. We multiply the solution at $y = 0$ by $\sin(\pi m x/a)$ and integrate:

$$\int_0^a \mathrm{d}x V \sin\left(\frac{\pi m x}{a}\right) = \sum_{n=1}^{\infty} C_n \int_0^a \mathrm{d}x \sin\left(\frac{\pi m x}{a}\right) \sin\left(\frac{\pi n}{a} x\right). \tag{3}$$

Let us prove that

$$\int_0^\alpha \sin(n\pi z/\alpha) \sin(m\pi z/\alpha)\,\mathrm{d}z = \begin{cases} 0, & \text{if } m \neq n, \\ \frac{\alpha}{2}, & \text{if } m = n. \end{cases}$$

Using integration by parts, we have

$$\begin{aligned} I &= \int_0^\alpha \sin(n\pi z/\alpha) \sin(m\pi z/\alpha)\,\mathrm{d}z \\ &= -\frac{\alpha}{m\pi} \overbrace{\sin(n\pi z/\alpha) \cos(m\pi z/\alpha)\big|_0^\alpha}^{=0} + \frac{\alpha}{m\pi} \frac{n\pi}{\alpha} \int_0^\alpha \cos(n\pi z/\alpha) \cos(m\pi z/\alpha)\,\mathrm{d}z \\ &= \left(\frac{\alpha}{m\pi}\right)^2 \frac{n\pi}{\alpha} \left[\overbrace{\cos(n\pi z/\alpha) \sin(m\pi z/\alpha)\big|_0^\alpha}^{=0} + \frac{n\pi}{\alpha} \int_0^\alpha \sin(n\pi z/\alpha) \sin(m\pi z/\alpha)\,\mathrm{d}z\right] \\ &= \left(\frac{\alpha}{m\pi}\right)^2 \left(\frac{n\pi}{\alpha}\right)^2 I. \end{aligned} \tag{4}$$

We have found that

$$I = \frac{n}{m} I, \tag{5}$$

which is of course true for $n = m$. If $n \neq m$, the only possible solution to equation (5) is $I = 0$. Now all that remains is to compute the integral for $n = m$. In the first step of the integration by parts we have found

$$\int_0^\alpha \sin(n\pi z/\alpha) \sin(n\pi z/\alpha)\,\mathrm{d}z = \int_0^\alpha \cos(n\pi z/\alpha) \cos(n\pi z/\alpha)\,\mathrm{d}z,$$

which we can write as

$$2I = \int_0^\alpha \left[\sin\left(n\pi z/\alpha\right)\sin\left(n\pi z/\alpha\right) + \cos\left(n\pi z/\alpha\right)\cos\left(n\pi z/\alpha\right)\right]\mathrm{d}z$$
$$= \int_0^\alpha \cos\left(\left(n-n\right)\pi z/\alpha\right)\mathrm{d}z = \alpha.$$

In total,

$$I = \frac{\alpha}{2}\delta_{n,m}.$$

On the RHS of equation (3) we thus have

$$\sum_{n=1}^{\infty} C_n \int_0^a \mathrm{d}x \sin\left(\frac{\pi m x}{a}\right)\sin\left(\frac{\pi n}{a}x\right) = C_m\frac{a}{2},$$

and on the LHS we have

$$V\int_0^a \mathrm{d}x \sin\left(\frac{\pi m x}{a}\right) = \begin{cases} 0, & \text{if } m = 2,4,\ldots \text{ even,} \\ \frac{2aV}{\pi m} & \text{if } m = 1,3,\ldots \text{ odd.} \end{cases}$$

Therefore,

$$C_n = \begin{cases} 0, & \text{if } m = 2,4,\ldots \text{ even,} \\ \frac{4V}{\pi m} & \text{if } m = 1,3,\ldots \text{ odd,} \end{cases}$$

and we finally have the complete solution

$$\Phi\left(x,y\right) = \sum_{n=\mathrm{odd}} \frac{4V}{\pi n}\exp\left(-\frac{\pi n}{a}y\right)\sin\left(\frac{\pi n}{a}x\right)$$
$$\{n = 2k-1\} = \frac{4V}{\pi}\sum_{k=1}^{\infty}\frac{1}{2k-1}\exp\left(-\frac{\pi\left(2k-1\right)}{a}y\right)\sin\left(\frac{\pi\left(2k-1\right)}{a}x\right).$$

Multipole expansion

When measuring the potential at some distant point from the charge distribution, we gradually lose the ability to resolve the exact details of the distribution. The larger details (lower order terms) are much more apparent in this case - the total charge, the dipole moment, etc., as can be seen in the multipole expansion of the potential,

$$\begin{aligned}\Phi(\vec{r}) &= \int \frac{\rho(\vec{r}')\,\mathrm{d}^3\vec{r}'}{|\vec{r}-\vec{r}'|} \\ &= \underbrace{\frac{1}{r}\int \rho(\vec{r}')\,\mathrm{d}^3\vec{r}'}_{=q/r} + \underbrace{\frac{\vec{r}}{r^3}\cdot\int \rho(\vec{r}')\,\mathrm{d}^3\vec{r}'\vec{r}'}_{=\vec{r}\cdot\vec{p}/r^3} + \underbrace{\frac{1}{2r^3}\int \rho(\vec{r}')\,\mathrm{d}^3\vec{r}'\left[\frac{3(\vec{r}'\cdot\vec{r})}{r^2} - r'^2\right]}_{=r_i r_j Q_{ij}/2r^5} + \dots\end{aligned}$$

where q is the total charge, $\vec{p}$ is the dipole moment and Q_{ij} is the quadrupole moment tensor. In class you have seen that the components of the quadrupole tensor satisfy

$$Q_{ij} = Q_{ji}, \qquad \mathrm{Tr}(Q) = Q_{11} + Q_{22} + Q_{33} = 0.$$

It is evident that while the total charge does not depend on the choice of origin, both the dipole and quadrupole moments do.

Problem 4

Given a thin disk of radius a with uniform surface charge σ_0 in the $x-y$ plane, find the potential to third order using the multipole expansion (with respect to the origin).

Solution

The charge density is

$$\rho(r,\theta,z) = \sigma_0\delta(z)\,\Theta(a-r).$$

We calculate the total charge,

$$q = \int \rho(\vec{r}')\,\mathrm{d}^3\vec{r}' = \sigma_0 \iiint \delta(z')\,\Theta(a-r')\,r'\mathrm{d}r'\mathrm{d}\theta'\mathrm{d}z' = \sigma_0\pi a^2,$$

dipole moment,

$$\begin{aligned}\vec{p} &= \int \rho\left(\vec{r}'\right) \mathrm{d}^3\vec{r}'\vec{r}' = \left(\int \rho\left(\vec{r}'\right) \mathrm{d}^3\vec{r}' x', \int \rho\left(\vec{r}'\right) \mathrm{d}^3\vec{r}' y', \int \rho\left(\vec{r}'\right) \mathrm{d}^3\vec{r}' z'\right) \\ &= \sigma_0 \iiint \delta\left(z'\right) \Theta\left(a - r'\right) r' \mathrm{d}r' \mathrm{d}\theta' \mathrm{d}z' \left(r' \sin\theta', r'\cos\theta', z'\right) \\ &= \sigma_0 \frac{R^3}{3}\left(0,0,0\right) = \vec{0},\end{aligned}$$

and quadrupole tensor $Q_{ij} = \int \left(3x'_i x'_j - r'^2\delta_{ij}\right) \rho\left(\vec{r}'\right) \mathrm{d}^3\vec{r}'$,

$$\begin{aligned}Q_{11} &= \int \left(3x'^2 - \left(x'^2 + y'^2 + z'^2\right)\right) \rho\left(\vec{r}'\right) \mathrm{d}^3\vec{r}' \\ &= \sigma_0 \iiint \left(3r'^2\cos^2\theta' - r'^2 - z^2\right) \delta\left(z'\right) \Theta\left(a - r'\right) r' \mathrm{d}r' \mathrm{d}\theta' \mathrm{d}z' \\ &= \sigma_0 \int_0^{2\pi} \int_0^a \left(3r'^2\cos^2\theta' - r'^2\right) r' \mathrm{d}r' \mathrm{d}\theta' \\ &= \sigma_0 \frac{a^4}{4} \int_0^{2\pi} \left(3\cos^2\theta' - 1\right) \mathrm{d}\theta' \\ &= \frac{\pi a^4 \sigma_0}{4}.\end{aligned}$$

Due to symmetry in the $x-y$ axis, $Q_{11} = Q_{22}$. We can obtain Q_{33} by using the trace symmetry,

$$Q_{33} = -\left(Q_{11} + Q_{22}\right) = -2Q_{11} = -\frac{\pi a^4 \sigma_0}{2}.$$

We could also have computed it explicitly,

$$\begin{aligned}Q_{33} &= \int \left(3z'^2 - \left(x'^2 + y'^2 + z'^2\right)\right) \rho\left(\vec{r}'\right) \mathrm{d}^3\vec{r}' \\ &= \sigma_0 \iiint \left(2z'^2 - r'^2\right) \delta\left(z'\right) \Theta\left(a - r'\right) r' \mathrm{d}r' \mathrm{d}\theta' \mathrm{d}z' \\ &= \sigma_0 \int_0^{2\pi} \int_0^a \left(-r'^2\right) r' \mathrm{d}r' \mathrm{d}\theta' \\ &= -\sigma_0 \frac{2\pi a^4}{4} \\ &= -\frac{\pi a^4 \sigma_0}{2}.\end{aligned}$$

When $i \neq j$, $Q_{ij} = 0$: the off-diagonal component Q_{xy} vanishes as follows,

$$\begin{aligned} Q_{xy} &= \sigma_0 \int 3x'y'\delta\left(z\right)\Theta\left(a-r\right)\mathrm{d}x'\mathrm{d}y'\mathrm{d}z' \\ &= \sigma_0 3 \int_0^a r'^3 \mathrm{d}r' \underbrace{\int_0^{2\pi} \overbrace{\cos\theta'\sin\theta'}^{=\sin(2\theta)/2}\mathrm{d}\theta'}_{=0}, \end{aligned}$$

and in the $\hat{z}$ direction, the integral vanishes since the delta function imposes $z = 0$. We therefore have the potential to third order,

$$\Phi\left(\vec{r}\right) = \frac{\sigma_0 \pi a^2}{r} + \frac{\pi a^4 \sigma_0}{4}\frac{\left(x^2+y^2-2z^2\right)}{2r^5}.$$