Homework 3

Question 1

Given the charge density $\rho = \rho_0 \sin(k_x x) \sin(k_y y) \sin(k_z z)$, find the potential when

1. The charge density is spread throughout the entire space.
2. The charge density is trapped inside a region $|z| < a$.

Question 2

Consider two parallel, grounded planes, one at $y = 0$ and the other at $y = d$. At $x = 0$ there is another charged plane, with surface charge $\sigma(y)$ (no z dependence). The potential vanishes at $x \to \pm\infty$.

1. Find the potential in the region $0 < y < d$, but leave the coefficients in integral form.
2. Now, instead of the plane with $\sigma(y)$, there is an infinite wire with line charge density λ located along the z axis at a distance $a < d$ from the bottom plane. Find the surface charge density $\sigma(y)$ which describes the wire, then solve for the coefficients you found in the previous subsection.
3. For the setup with the wire, find the surface charge density on both planes (in the form of an expansion in the Laplace eigenfunctions you previously found).

Question 3

An infinite plane is charged with $\sigma(x, y) = \sigma_0 \sin(ax + by)$. Find $\Phi(\vec{r})$. Does $\Phi(z \to \pm\infty) = 0$?

Question 4

Expand the Green's function of the Laplacian in spherical harmonics, and show that it takes the form

$$\frac{1}{|\vec{r}-\vec{r}'|} = \sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell}\frac{1}{2\ell+1}\frac{r_<^{\ell}}{r_>^{\ell+1}}Y_{\ell m}^{*}\left(\theta',\varphi'\right)Y_{\ell m}\left(\theta,\varphi\right), \tag{1}$$

where

$$r_> = \max\left(r',r\right), \quad r_> = \min\left(r',r\right).$$

Guidance Recall from class that due to completeness and orthogonality of the basis $Y_{\ell,m}$, you can write

$$\delta^{(3)}\left(\vec{r}-\vec{r}'\right) = \frac{1}{r^2}\delta\left(r-r'\right)\sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell}Y_{\ell m}^{*}\left(\theta',\varphi'\right)Y_{\ell m}\left(\theta,\varphi\right).$$

Solve the Green's function equation

$$\nabla^2 G = \delta^{(3)}\left(\vec{r}-\vec{r}'\right),$$

by using spherical separation of variables of the form

$$G\left(\vec{r},\vec{r}'\right) = \sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell}R_{\ell}\left(r,r'\right)Y_{\ell m}^{*}\left(\theta',\varphi'\right)Y_{\ell m}\left(\theta,\varphi\right),$$

and find the radial part in each of the regions $r > r'$ and $r < r'$. Find the matching conditions at $r = r'$, and use them to obtain the solution (1).