# Homework 3

## Question 1

Given the charge density $\rho = \rho_0 \sin(k_x x)\sin(k_y y)\sin(k_z z)$, find the potential when

1. The charge density is spread throughout the entire space.
2. The charge density is trapped inside a region $|z| < a$.

## Solution

1. The equation to solve is the Poisson equation,

$$\nabla^2 \Phi = -4\pi\rho_0 \sin(k_x x)\sin(k_y y)\sin(k_z z).$$

The Cartesian spectral decomposition of the Laplacian includes the functions $\sin(x), \cos(x), e^{-x}, e^{+x}$. In this case we guess (based on the form of the charge density) that the solution is made out of sines only,

$$\Phi(x,y,z) = \sum_{m,n,l} A_{n,m,l} \sin(nx)\sin(my)\sin(lz).$$

Plugging this solution into the Poisson equation, we have

$$-\sum_{m,n,l} A_{n,m,l}\left(n^2+m^2+l^2\right)\sin(nx)\sin(my)\sin(lz) = -4\pi\rho_0 \sin(k_x x)\sin(k_y y)\sin(k_z z).$$

Therefore, by the orthogonality of the sine basis, $n = k_x$, $m = k_y$ and $l = k_z$, with

$$A = \frac{4\pi\rho_0}{k_x^2+k_y^2+k_z^2}.$$

The potential is therefore

$$\Phi(x,y,z) = \frac{4\pi\rho_0}{k_x^2+k_y^2+k_z^2}\sin(k_x x)\sin(k_y y)\sin(k_z z).$$

2. In this case, the charge density is bounded in $z$, so we have

$$\rho = \rho_0 \sin(k_x x) \sin(k_y y) \sin(k_z z) \Theta\left(a^2 - z^2\right).$$

Since in $x, y$ the problem is the same, we guess a solution of the form

$$\Phi \sim \sin(k_x x) \sin(k_y y) \psi(z),$$

where the function $\psi(z)$ satisfies either the Laplace or Poisson equations, depending on the region:

$$\psi'' - k^2 \psi = \begin{cases} 0 & |z| > a \\ -4\pi\rho_0 \sin(k_z z) & |z| < a \end{cases}$$

For $|z| > a$ and $k \in \mathbb{R}$ the solution is

$$\psi(z) = Ae^{-kz} + Be^{kz},$$

where $k^2 = k_x^2 + k_y^2$, and since $\Phi(z \to \pm\infty) = 0$ we have

$$\psi(z > a) = Ae^{-kz}, \ \psi(z < -a) = Be^{kz}.$$

For $|z| < a$ we take the solution to be a linear combination of the entire basis of solutions to the Homogeneous (Laplace) equation and add the particular solution,

$$\psi(|z| < a) = C\sin(kz) + D_1 e^{kz} + D_2 e^{-kz}.$$

From subsection (1) we know that the particular solution $C\sin(kz)$ has the coefficient

$$C = \frac{4\pi\rho_0}{k_x^2 + k_y^2 + k_z^2}.$$

From continuity across $z = \pm a$ of both the potential and the first derivative[1] we find

$$\begin{aligned}
Ae^{-ka} &= \frac{4\pi\rho_0}{k_x^2+k_y^2+k_z^2}\sin(ka) + D_1e^{ka} + D_2e^{-ka},\\
Be^{-ka} &= \frac{4\pi\rho_0}{k_x^2+k_y^2+k_z^2}\sin(-ka) + D_1e^{-ka} + D_2e^{ka},\\
-kAe^{-ka} &= k\left[\frac{4\pi\rho_0}{k_x^2+k_y^2+k_z^2}\cos(ka) + D_1e^{ka} - D_2e^{-ka}\right],\\
kBe^{-ka} &= k\left[\frac{4\pi\rho_0}{k_x^2+k_y^2+k_z^2}\cos(-ka) + D_1e^{-ka} - D_2e^{ka}\right],
\end{aligned}$$

$$\implies D_1 = -\frac{2\pi\rho_0 e^{-ka}}{k_x^2+k_y^2+k_z^2}\left(\sin(ka)+\cos(ka)\right) = -e^{-2ka}D_2,$$

$$\implies A = \frac{4\pi\rho_0 e^{ka}}{k_x^2+k_y^2+k_z^2}\sin(ka) = -B.$$

The solution is therefore

$$\Phi(x,y,z) = \frac{2\pi\rho_0\sin(k_x x)\sin(k_y y)}{k_x^2+k_y^2+k_z^2}\times\begin{cases} 2e^{ka}\sin(ka)\,e^{-kz}, & z>a\\ 2\sin(kz) + e^{ka}\left(\sin(ka)+\cos(ka)\right)\left(e^{-kz}-e^{-2ka}e^{kz}\right), & |z|<a\\ -2e^{ka}\sin(ka)\,e^{kz}, & z<-a\end{cases}$$

[1]Note that there is no jump in the electric field since there is no delta function in the charge density!

# Question 2

Consider two parallel, grounded planes, one at $y = 0$ and the other at $y = d$. At $x = 0$ there is another charged plane, with surface charge $\sigma(y)$ (no $z$ dependence). The potential vanishes at $x \to \pm\infty$.

1. Find the potential in the region $0 < y < d$, but leave the coefficients in integral form.
2. Now, instead of the plane with $\sigma(y)$, there is an infinite wire with line charge density $\lambda$ located along the $z$ axis at a distance $a < d$ from the bottom plane. Find the surface charge density $\sigma(y)$ which describes the wire, then solve for the coefficients you found in the previous subsection.
3. For the setup with the wire, find the surface charge density on both planes (in the form of an expansion in the Laplace eigenfunctions you previously found).

# Solution

1. We separate the problem into two regions in which the Laplace equations is satisfied,

$$x > 0, x < 0.$$

Per each region, we have the solution we saw in class (Q3 in class 3)

$$\Phi_{\mathrm{I/II}}(x, y) = \sum_{n=1}^{\infty} \left[A_n^{\mathrm{I/II}} \exp\left(-\frac{n\pi x}{d}\right) + B_n^{\mathrm{I/II}} \exp\left(+\frac{n\pi x}{d}\right)\right] \sin\left(\frac{n\pi y}{d}\right),$$

where $x < 0$ is denoted by I and $x > 0$ by II. Since $\Phi(x \to \pm\infty) = 0$, $A_n^{\mathrm{I}} = B_n^{\mathrm{II}} = 0$ and thus

$$\Phi(x, y) = \begin{cases} \sum_{n=1}^{\infty} A_n^{\mathrm{II}} \exp\left(-\frac{n\pi x}{d}\right) \sin\left(\frac{n\pi y}{d}\right) & \text{, if } x > 0 \\ \sum_{n=1}^{\infty} B_n^{\mathrm{I}} \exp\left(+\frac{n\pi x}{d}\right) \sin\left(\frac{n\pi y}{d}\right) & \text{, if } x < 0. \end{cases}$$

The potential is continuous across $x = 0$ and therefore $A_n^{\mathrm{II}} = B_n^{\mathrm{I}} \equiv A_n$, and so

$$\Phi(x, y) = \sum_{n=1}^{\infty} A_n \exp\left(-\frac{n\pi |x|}{d}\right) \sin\left(\frac{n\pi y}{d}\right).$$

The jump in the electric field across $x = 0$ is then

$$4\pi\sigma = -\left.\frac{\partial\Phi}{\partial x}\right|_{x=0^+} + \left.\frac{\partial\Phi}{\partial x}\right|_{x=0^-} = \sum_{n=1}^{\infty} \frac{2\pi n}{d} A_n \sin\left(\frac{\pi n y}{d}\right),$$

so we have

$$\frac{2\pi n}{d}A_n = \frac{2}{d}\int\limits_0^d \mathrm{d}y 4\pi\sigma(y)\sin\left(\frac{\pi n y}{d}\right)$$
$$\implies A_n = \frac{4}{n}\int\limits_0^d \mathrm{d}y 4\pi\sigma(y)\sin\left(\frac{\pi n y}{d}\right).$$

**Sanity check**: the potential has the form $\Phi \sim \exp\left(-\frac{n\pi|x|}{d}\right)$. Therefore the Laplacian has a term of the form

$$\frac{d^2}{dx^2}\exp\left(-\frac{n\pi|x|}{d}\right) = -\frac{n\pi}{d}\frac{d^2}{dx^2}|x|\exp\left(-\frac{n\pi|x|}{d}\right) + \left(\frac{n\pi}{d}\frac{d}{dx}|x|\right)^2\exp\left(-\frac{n\pi|x|}{d}\right),$$

where

$$\frac{d}{dx}|x| = \frac{x}{|x|}, \frac{d^2}{dx^2}|x| = \frac{1}{|x|} - \frac{x^2}{|x|^3}.$$

Therefore,

$$\nabla^2\Phi = \begin{cases} 0 & x > 0 \\ \infty & x = 0 \\ 0 & x < 0 \end{cases} = \delta(x)$$

as expected for the setup of a charged plane at $x = 0$.

2. The appropriate charge density is $\sigma(y) = \lambda\delta(y-a)$, and thus

$$A_n = \frac{4}{n}\int\limits_0^d \mathrm{d}y\,\lambda\delta(y-a)\sin\left(\frac{\pi n y}{d}\right) = \frac{4\lambda}{n}\sin\left(\frac{\pi n a}{d}\right),$$

and the potential is

$$\Phi(x,y) = \sum_{n=1}^{\infty}\frac{4\lambda}{n}\sin\left(\frac{\pi n a}{d}\right)\sin\left(\frac{\pi n y}{d}\right)\exp\left(-\frac{\pi n|x|}{d}\right).$$

3. The charge density on the upper plane is

$$\sigma_d = \frac{1}{4\pi}\frac{\partial}{\partial y}\Phi(y=d) = \frac{\lambda}{d}\sum_{n=1}^{\infty}(-1)^n\sin\left(\frac{\pi n a}{d}\right)\exp\left(-\frac{\pi n|x|}{d}\right),$$

and on the lower plane

$$\sigma_0 = -\frac{1}{4\pi}\frac{\partial}{\partial y}\Phi(y=0) = -\frac{\lambda}{d}\sum_{n=1}^{\infty}\sin\left(\frac{\pi n a}{d}\right)\exp\left(-\frac{\pi n|x|}{d}\right).$$

## Question 3

An infinite plane is charged with $\sigma(x,y) = \sigma_0 \sin(ax+by)$. Find $\Phi(\vec{r})$. Does $\Phi(z \to \pm\infty) = 0$?

## Solution

We write the charge density

$$\rho(x,y,z) = \sigma_0 \sin(ax+by)\,\delta(z),$$

such that the potential solves

$$\nabla^2\Phi = -4\pi\rho(x,y,z).$$

For $z \neq 0$, $\nabla^2\Phi = 0$, i.e.,

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)\Phi = 0.$$

This suggests using the solution

$$\Phi(x,y,z) = X(x)\,Y(y)\,Z(z),$$

which when substituted back into the Poisson equation gives

$$\frac{1}{X}\frac{d^2X}{dx^2} + \frac{1}{Y}\frac{d^2Y}{dy^2} + \frac{1}{Z}\frac{d^2Z}{dz^2} = 0.$$

Since each piece only depends on one variable, the only possible solution is that each piece solves an equation of the form

$$\frac{1}{X}\frac{d^2X}{dx^2} = a_x, \qquad \frac{1}{Y}\frac{d^2Y}{dy^2} = a_y, \qquad \frac{1}{Z}\frac{d^2Z}{dz^2} = a_z,$$

where $a_x, a_y, a_z$ are constants which solve $a_x + a_y + a_z = 0$.

It is very important to note that in this case $\Phi(z \to \infty) = \Phi(z \to -\infty) = 0$. This is unlike the case of an infinite plane with constant surface charge density, where you have seen that far away from the plane, the potential is non-zero!

However, the sine boundary condition on the slab means that the total charge (averaged) is 0, since there is an equal amount of negative and positive charge on the slab. From Gauss' law, we find that the potential away from the slab must vanish.

In this case, $Z(z)$ must be a decaying solution - namely, $a_z \equiv k^2 > 0$:

$$\frac{1}{Z}\frac{d^2Z}{dz^2} = a_z = k^2 \implies Z(z) = Ae^{kz} + Be^{-kz}.$$

To satisfy the vanishing of the potential at $z \to \pm\infty$, $A = 0$ for $z > 0$ and $B = 0$ for $z < 0$. The boundary condition on the slab is the continuity of $\Phi$, which gives

$$Z\left(0^{+}\right)=Z\left(0^{-}\right) \implies A=B.$$

In addition, on the slab we have the Poisson equation

$$\nabla^{2}\Phi=-4\pi\sigma\left(x,y\right)\delta\left(z\right).$$

We can integrate it and obtain an jump condition,

$$\int_{0^{-}}^{0^{+}}\mathrm{d}z\frac{d^{2}}{dz^{2}}\Phi=XY\left(\left.\frac{dZ}{dz}\right|_{0^{+}}-\left.\frac{dZ}{dz}\right|_{0^{-}}\right)=-4\pi\sigma\left(x,y\right).$$

Plugging in our solution, we obtain

$$XYA\left(-ke^{-k0}-ke^{k0}\right)=-4\pi\sigma\left(x,y\right),$$

and thus

$$X\left(x\right)Y\left(y\right)=\frac{2\pi\sigma\left(x,y\right)}{kA}.$$

Lastly, we compute $k$ by using the Laplace equation,

$$\frac{1}{XY}\left(Y\frac{d^{2}X}{dx^{2}}+X\frac{d^{2}Y}{dy^{2}}\right)+\frac{1}{Z}\frac{d^{2}Z}{dz^{2}}=\frac{1}{XY}\left(\frac{d^{2}\left(XY\right)}{dx^{2}}+\frac{d^{2}\left(XY\right)}{dy^{2}}\right)+\frac{1}{Z}\frac{d^{2}Z}{dz^{2}}=-a^{2}-b^{2}+k^{2}=0.$$

We therefore have

$$\Phi\left(x,y,z\right)=\frac{2\pi}{\sqrt{a^{2}+b^{2}}}\sigma_{0}\sin\left(ax+by\right)\begin{cases}e^{-\sqrt{a^{2}+b^{2}}z} & z>0,\\ e^{\sqrt{a^{2}+b^{2}}z} & z<0.\end{cases}$$

# Question 4

Expand the Green's function of the Laplacian in spherical harmonics, and show that it takes the form

$$\frac{1}{|\vec{r}-\vec{r}'|}=\sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell}\frac{1}{2\ell+1}\frac{r_<^{\ell}}{r_>^{\ell+1}}Y_{\ell m}^*\left(\theta',\varphi'\right)Y_{\ell m}\left(\theta,\varphi\right), \tag{1}$$

where

$$r_> = \max\left(r',r\right), \quad r_> = \min\left(r',r\right).$$

**Guidance** Recall from class that due to completeness and orthogonality of the basis $Y_{\ell,m}$, you can write

$$\delta^{(3)}\left(\vec{r}-\vec{r}'\right)=\frac{1}{r^2}\delta\left(r-r'\right)\sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell}Y_{\ell m}^*\left(\theta',\varphi'\right)Y_{\ell m}\left(\theta,\varphi\right).$$

Solve the Green's function equation

$$\nabla^2 G=\delta^{(3)}\left(\vec{r}-\vec{r}'\right),$$

by using spherical separation of variables of the form

$$G\left(\vec{r},\vec{r}'\right)=\sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell}R_\ell\left(r,r'\right)Y_{\ell m}^*\left(\theta',\varphi'\right)Y_{\ell m}\left(\theta,\varphi\right),$$

and find the radial part in each of the regions $r>r'$ and $r<r'$. Find the matching conditions at $r=r'$, and use them to obtain the solution (1).

# Solution

We guess the separable solution of the Green's function, insert it into the spherical Laplacian and obtain

$$Y_{\ell m}^*Y_{\ell m}\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d}{dr}R_\ell\right)+R_\ell Y_{\ell m}^*\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\left(Y_{\ell m}\right)\right)+\frac{1}{\sin^2\theta}\frac{\partial}{\partial\varphi}\left(Y_{\ell m}\right)\right]=\frac{1}{r^2}\delta\left(r-r'\right)Y_{\ell m}^*\left(\theta',\varphi'\right)Y_{\ell m}\left(\theta,\varphi\right),$$

where we have noted that only $Y_{\ell m}$ depends on $\theta,\varphi$ ($Y_{\ell m}^*=Y_{\ell m}^*\left(\theta',\varphi'\right)$!). Recall that by definition,

$$\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}Y_{\ell m}\right)+\frac{1}{\sin^2\theta}\frac{\partial}{\partial\varphi}Y_{\ell m}=-\frac{\ell\left(\ell+1\right)}{r^2}Y_{\ell m},$$

and thus after dividing by $Y_{\ell m}^*Y_{\ell m}$ we obtain the radial equation

$$\left[\frac{d}{dr}\left(r^2\frac{d}{dr}R_\ell\right)-\ell\left(\ell+1\right)R_\ell\right]R_\ell=\delta\left(r-r'\right).$$

Guessing a polynomial form of the solution $R_\ell = Cr^\alpha$ for any $r \neq r'$, we obtain

$$\begin{aligned} \frac{d}{dr} r^2 \frac{d}{dr} r^\alpha &= \ell\left(\ell+1\right) r^\alpha \\ \Longrightarrow \alpha\left(\alpha+1\right) &= \ell\left(\ell+1\right) \\ \Longrightarrow \alpha &= \ell, -\left(\ell+1\right). \end{aligned}$$

Therefore, to obtain regular solutions when $r \to 0, \infty$, the solution is

$$R_\ell\left(r, r'\right) = \begin{cases} \frac{A_\ell}{r^{\ell+1}}, & 0 < r' < r, \\ B_\ell r^\ell, & 0 < r < r'. \end{cases}$$

We use continuity of $R_\ell$, as well as the jump of its first derivative (due to the delta-function) to obtain the matching conditions

$$\begin{aligned} R_\ell\left(r \to r'^{+}\right) &= R_\ell\left(r \to r'^{-}\right), \\ r^2 \frac{d}{dr} R_\ell \bigg|_{r \to r'+} - r^2 \frac{d}{dr} R_\ell \bigg|_{r \to r'-} &= 1, \end{aligned}$$

and find that

$$\begin{aligned} A_\ell &= B_\ell \left(r'\right)^{2\ell+1}, \\ B_\ell \left(r'\right)^{\ell+1} &= -\frac{1}{2\ell+1}. \end{aligned}$$

Therefore,

$$R_\ell\left(r, r'\right) = -\frac{1}{2\ell+1} \begin{cases} \frac{\left(r'\right)^\ell}{r^{\ell+1}}, & 0 < r' < r, \\ \frac{r^\ell}{\left(r'\right)^{\ell+1}}, & 0 < r < r', \end{cases}$$

and with the definitions $r_> = \max\left(r', r\right), \quad r_> = \min\left(r', r\right)$ we have the complete expansion,

$$\frac{1}{\left|\vec{r} - \vec{r}'\right|} = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \frac{1}{2\ell+1} \frac{r_<^\ell}{r_>^{\ell+1}} Y_{\ell m}^*\left(\theta', \varphi'\right) Y_{\ell m}\left(\theta, \varphi\right).$$