Class Exercise 4 - Spherical separation of variables

Spherical separation of variables

The main steps in developing the spherical separation of variables:

- Laplace equation in spherical coordinates

$$0 = \nabla^2 \Phi = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial}{\partial r}\Phi\right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial \Phi}{\partial \theta}\right) + \frac{1}{r^2 \sin^2\theta}\frac{\partial^2 \Phi}{\partial \varphi^2},$$

- Look for a solution $\Phi = R\left(r\right) Y\left(\theta, \varphi\right)$ and obtain

$$\underbrace{\frac{1}{R}\frac{\mathrm{d}}{\mathrm{d}r}\left(r^2 \frac{\mathrm{d}R}{\mathrm{d}r}\right)}_{\text{only } r \text{ dependent}} + \underbrace{\frac{1}{Y}\left[\frac{1}{\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial Y}{\partial \theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2 Y}{\partial \varphi^2}\right]}_{\text{only } \theta, \varphi \text{ dependent}} = 0.$$

- Set the angular constant to be $\ell\left(\ell+1\right)$ and find for the radial part

$$\frac{\mathrm{d}}{\mathrm{d}r}\left(r^2 \frac{\mathrm{d}R}{\mathrm{d}r}\right) = \ell(\ell+1)R,$$

  with the solution $R_\ell\left(r\right) = Ar^\ell + Br^{-(\ell+1)}$.

- For $Y\left(\theta, \varphi\right) = P\left(\theta\right) Q\left(\varphi\right)$, the angular Laplacian becomes

$$\underbrace{\frac{1}{P}\sin\theta \frac{\mathrm{d}}{\mathrm{d}\theta}\left(\sin\theta \frac{\mathrm{d}P}{\mathrm{d}\theta}\right) + \ell(\ell+1)\sin^2\theta}_{\text{only } \theta \text{ dependence}} + \underbrace{\frac{1}{Q}\frac{\partial^2 Q}{\partial \varphi^2}}_{\varphi \text{ dependence}} = 0,$$

  and we choose

$$\frac{1}{Q}\frac{\partial^2 Q}{\partial \varphi^2} = -m^2,$$

  so $Q_m\left(\varphi\right) = Ce^{im\varphi} + De^{-im\varphi}$. We are left with a differential equation for P, where it is convenient to write $P = P\left(\cos\theta\right)$.

- The functions $P_\ell^m(\cos\theta)$ are the associated Legendre polynomials, which solve the *general Legendre equation*

$$\frac{\mathrm{d}}{\mathrm{d}x}\left((1-x^2)\frac{\mathrm{d}P_\ell^m}{\mathrm{d}x}\right)+\left\{\ell(\ell+1)+\frac{m^2}{1-x^2}\right\}P_\ell^m=0,$$

  and are related to the Legendre polynomials $P_\ell^0(x) := P_\ell(x)$ through

$$P_\ell^m(x)=(-1)^m\left(1-x^2\right)^{m/2}\frac{\mathrm{d}^m}{\mathrm{d}x^m}P_\ell(x)\,,\quad m>0,$$
$$P_\ell^{-m}(x)=(-1)^m\frac{(\ell-m)!}{(\ell+m)!}P_\ell^m(x)\,.$$

- In class, you found the possible values of ℓ by determining that the polynomial solution for P can only terminate (and thus not diverge for $\cos\theta=\pm1$) for integer ℓ. The possible values for m are in the range $-\ell\le m\le\ell$.

- The normalized spherical harmonics $Y_{\ell m}(\theta,\varphi)$ are therefore given by

$$Y_{\ell m}(\theta,\varphi)=\sqrt{\frac{(2\ell+1)\,(\ell-m)!}{4\pi\,(\ell+m)!}}P_\ell(\cos\theta)\,e^{im\varphi},$$

  such that

$$\int_0^{2\pi}\mathrm{d}\varphi\int_0^{\pi}\sin\theta\,\mathrm{d}\theta Y^*_{\ell' m'}(\theta,\varphi)\,Y_{\ell m}(\theta,\varphi)=\delta_{\ell\ell'}\delta_{mm'},$$

  and

$$\sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell}Y^*_{\ell m}(\theta,\varphi)\,Y_{\ell m}(\theta',\varphi')=\delta(\varphi-\varphi')\,\delta(\cos\theta-\cos\theta')\,.$$

We find the spherical spectral decomposition

$$\Phi=\sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell}\left(\underbrace{A_{\ell,m}r^{\ell}}_{\text{valid inside some sphere}}+\underbrace{B_{\ell,m}r^{-\ell-1}}_{\text{valid outside some sphere}}\right)Y_{\ell m}(\theta,\varphi)\,.$$

Azimuthal symmetry

- In this case the problem is symmetric under $\varphi\to\varphi+\varphi_0$, or $\partial\Phi/\partial\varphi=0$, and thus $m=0$.

- The Legendre polynomials $P_\ell(x)$ are degree ℓ polynomials in $x=\cos\theta$ which satisfy the *Legendre differential equation*

$$\frac{\mathrm{d}}{\mathrm{d}x}\left((1-x^2)\frac{\mathrm{d}P}{\mathrm{d}x}\right)+\ell(\ell+1)P=0,$$

and are normalized such that $P_\ell(\cos\theta = 1) = 1$. A solution with azimuthal symmetry will be of the form

$$\Phi = \sum_{\ell=0}^{\infty}\left(\underbrace{A_\ell r^\ell}_{r\in[0,R]} + \underbrace{B_\ell r^{-\ell-1}}_{r\in[R,\infty)}\right) P_\ell(\cos\theta)\,.$$

- Orthogonality:

$$\int_{-1}^{1} \mathrm{d}x\, P_\ell(x)P_{\ell'}(x) = \int_{0}^{\pi} \sin\theta\,\mathrm{d}\theta\, P_\ell(\cos\theta)P_{\ell'}(\cos\theta) = \frac{2}{2\ell+1}\delta_{\ell\ell'},$$

- Completeness:

$$\sum_{\ell=0}^{\infty}\frac{2\ell+1}{2}P_\ell(x)P_\ell(y) = \delta(x-y), \qquad |x|,|y| \le 1.$$

- The Rodriguez formula

$$P_\ell(x) = \frac{1}{2^\ell \ell!}\frac{\mathrm{d}^\ell}{\mathrm{d}x^\ell}\left(x^2-1\right)^\ell, \qquad \ell = 0,1,2,\ldots$$

allows to compute the polynomials explicitly, for example:

$$\begin{aligned} P_0(x) &= 1 \\ P_1(x) &= x \\ P_2(x) &= \frac{1}{2}\left(3x^2-1\right) \\ P_3(x) &= \frac{1}{2}\left(5x^3-3x\right) \\ &\vdots \end{aligned}$$

- $P_\ell(x) = (-1)^\ell P_\ell(-x)$.

Problem 1

A conducting sphere of radius R is placed in a constant field $\vec{E}_0$. Find the charge distribution on the sphere.

Solution

The field inside a conductor vanishes, so from Gauss the charge density for $r < R$ is zero,

$$\vec{\nabla} \cdot \vec{E} = 4\pi\rho = 0 \implies \rho = 0 \text{ for } r < R.$$

So if there is a charge density, it is concentrated entirely on the boundary $r = R$. We have the boundary condition

$$E_r\left(R^+\right) - \underbrace{E_r\left(R^-\right)}_{=0} = 4\pi\sigma,$$

and therefore need only to find $E_r\left(R^+\right)$. The potential satisfies the Laplace equation everywhere apart from the surface of the sphere (since $\rho = 0$ for $r < R$ and $r > R$). With the electric field vanishing, it is simply a constant inside the sphere, and we may set that constant to zero,

$$\Phi\left(r < R\right) = 0.$$

For $r > R$, we rotate the axes such that $\vec{E}_0 = E_0\hat{z}$. Therefore, far away from the sphere, we must have

$$\Phi\left(r \gg R\right) = -E_0 z + C = -E_0 r\cos\theta + C,$$

where we can set $C = 0$. The external field breaks the spherical symmetry, but does not ruin the symmetry in φ. We may therefore choose

$$\Phi\left(r, \theta\right) = R\left(r\right)\Theta\left(\theta\right),$$

and obtain the Laplace equation

$$\underbrace{\frac{1}{R}\frac{\mathrm{d}}{\mathrm{d}r}\left(r^2\frac{\mathrm{d}R}{\mathrm{d}r}\right)}_{\ell(\ell+1)} + \underbrace{\frac{1}{\Theta\sin\theta}\frac{\mathrm{d}}{\mathrm{d}\theta}\left(\sin\theta\frac{\mathrm{d}\Theta}{\mathrm{d}\theta}\right)}_{-\ell(\ell+1)} = 0.$$

We separate the equations,

$$\frac{\mathrm{d}^2 R}{\mathrm{d}r^2} + \frac{2}{r}\frac{\mathrm{d}R}{\mathrm{d}r} - \frac{\ell\left(\ell+1\right)}{r^2}R = 0,$$
$$\frac{\mathrm{d}^2\Theta}{\mathrm{d}\theta^2} + \cot\theta\frac{\mathrm{d}\Theta}{\mathrm{d}\theta} + \ell\left(\ell+1\right)\Theta = 0,$$

and find the corresponding solutions,

$$R\left(r\right) = Ar^{\ell} + Br^{-(\ell+1)},$$
$$\Theta\left(\theta\right) = P_\ell\left(\cos\theta\right), \ell = 0, 1, 2, \ldots$$

Therefore,

$$\Phi\left(r,\theta\right) = \begin{cases} 0 & r < R \\ \sum_{\ell=0}^{\infty}\left(A_\ell r^\ell + B_\ell r^{-(\ell+1)}\right)P_\ell\left(\cos\theta\right) & r > R \end{cases}$$

and from continuity across $r = R$,

$$\Phi\left(R^-\right) = \Phi\left(R^+\right)$$
$$\Longrightarrow \sum_{\ell=0}^{\infty}\left(A_\ell R^\ell + B_\ell R^{-(\ell+1)}\right)P_\ell\left(\cos\theta\right) = 0.$$

We use the orthogonal property of the Legendre polynomials by multiplying with $P_{\ell'}$ and integrating,

$$\sum_{\ell=0}^{\infty}\left(A_\ell R^\ell + B_\ell R^{-(\ell+1)}\right)\int_{-1}^{1}\mathrm{d}x\, P_\ell(x)P_{\ell'}(x) = \sum_{\ell=0}^{\infty}\left(A_\ell R^\ell + B_\ell R^{-(\ell+1)}\right)\frac{2}{2\ell+1}\delta_{\ell\ell'}$$
$$= \left(A_{\ell'}R^{\ell'} + B_{\ell'}R^{-(\ell'+1)}\right)\frac{2}{2\ell'+1} = 0.$$

We obtain $A_{\ell'} = -B_{\ell'}R^{-(2\ell'+1)}$ for $\ell' = 0, 1, 2\ldots$.

For $\ell = 1$ we have $P_1\left(\cos\theta\right) = \cos\theta$ and therefore we can write the first few terms of the expansion in the limit of large r,

$$\Phi\left(r \gg R, \theta\right) = A_0 + A_1 r\cos\theta + \sum_{\ell=2}^{\infty}A_\ell r^\ell P_\ell\left(\cos\theta\right) = -E_0 r\cos\theta.$$

Hence, we need to have $A_1 = -E_0$, $A_{\ell\neq1} = 0$. That means $A_1R^3 = -B_1$ and $B_{\ell\neq1} = 0$ so that in total,

$$\Phi\left(r,\theta\right) = \begin{cases} 0 & r < R \\ -E_0\left(r - \frac{R^3}{r^2}\right)\cos\theta & r > R \end{cases}$$

We can now compute the charge density on the sphere from the jump in the field,

$$\vec{E}\left(r\right)=-\left(\frac{\partial\Phi}{\partial r}\hat{r}+\frac{1}{r}\frac{\partial\Phi}{\partial\theta}\hat{\theta}\right)=\begin{cases}0 & r<R\\ E_0\left(1+2\left(\frac{R}{r}\right)^3\right)\cos\theta\hat{r}-E_0\left(1-\left(\frac{R}{r}\right)^3\right)\sin\theta\hat{\theta} & r>R\end{cases}$$

Thus, the surface charge density is

$$\sigma=\frac{3E_0}{4\pi}\cos\theta.$$

Notice that in the equatorial plane $\theta=\pi/2$ it vanishes, and at $\theta=0,\pi$ the charge is most dense, which makes sense for a setup in which the field is directed from the south to the north pole of the sphere (creating a sort of "dipole" distribution of charge).

Problem 2

Two concentric spheres of radii a, b such that $a < b$, are held at potentials $V_1(\theta) = \alpha \cos\theta$, $V_2(\theta) = \beta \cos^3\theta$, respectively. Find the potential everywhere in space.

Solution

No charges given $\implies \rho(\vec{r}) = 0$. Additionally, the problem is symmetric in φ and thus we already know that the solution takes the form

$$\Phi(r,\theta) = \sum_{\ell=0}^{\infty} \left(A_\ell r^\ell + B_\ell r^{-(\ell+1)}\right) P_\ell(\cos\theta).$$

However, for each region (outside both spheres, between them and inside the smaller sphere) we will have different coefficients, so we write

$$\Phi(r,\theta) = \begin{cases} \sum_{\ell=0}^{\infty} A_\ell r^\ell P_\ell(\cos\theta) & r < a \\ \sum_{\ell=0}^{\infty} \left(B_\ell r^\ell + C_\ell r^{-(\ell+1)}\right) P_\ell(\cos\theta) & a < r < b \\ \sum_{\ell=0}^{\infty} D_\ell r^{-(\ell+1)} P_\ell(\cos\theta) & r > b \end{cases}$$

and sew the solutions together at the boundaries. At $r = a$ we find

$$\sum_{\ell=0}^{\infty} A_\ell a^\ell P_\ell(\cos\theta) = \alpha\cos\theta = \alpha P_1(\cos\theta) \implies A_1 = \alpha/a,$$

and therefore $A_{\ell\neq 1} = 0$. Similarly, at $r = b$ we find

$$\sum_{\ell=0}^{\infty} D_\ell b^{-(\ell+1)} P_\ell(\cos\theta) = \beta\cos^3\theta,$$

and since $P_3(\cos\theta) = (5\cos^3\theta - 3\cos\theta)/2$, we can write $\cos^3\theta = (2P_3(\cos\theta) + 3P_1\cos\theta)/5$ and equate coefficients to obtain

$$D_1 b^{-(1+1)} + D_3 b^{-(3+1)} = \beta\left(\frac{2}{5}P_3(\cos\theta) + \frac{3}{5}P_1\cos\theta\right)$$
$$\implies D_1 = \frac{3}{5}\beta b^2, D_3 = \frac{2}{5}\beta b^4, D_{\ell\neq 1,3} = 0.$$

Now, we use the continuity at the boundaries, beginning with $r = a$:

$$\sum_{\ell=0}^{\infty} \left(B_\ell a^\ell + C_\ell a^{-(\ell+1)}\right) P_\ell(\cos\theta) = \alpha\cos\theta,$$

therefore,

$$B_1 a + \frac{C_1}{a^2} = \alpha,$$
$$B_\ell a^\ell + C_\ell a^{-(\ell+1)} = 0, \quad \ell > 1.$$

At $r = b$,

$$\sum_{\ell=0}^{\infty} \left(B_\ell b^\ell + C_\ell b^{-(\ell+1)}\right) P_\ell(\cos\theta) = \beta \cos^3\theta = \frac{\beta}{5}\left(2P_3(\cos\theta) + 3P_1 \cos\theta\right),$$

and thus

$$B_1 b + \frac{C_1}{b^2} = \frac{3\beta}{5},$$
$$B_3 b^3 + \frac{C_3}{b^4} = \frac{2\beta}{5},$$
$$B_\ell b^\ell + C_\ell b^{-(\ell+1)} = 0, \quad \ell \neq 1, 3.$$

For $\ell \neq 1, 3$ we find $B_\ell = C_\ell = 0$. For $\ell = 3$,

$$B_3 = \frac{2\beta}{5b^3} - \frac{C_3}{b^7},$$
$$B_3 a^3 + \frac{C_3}{a^4} = \left(\frac{2\beta}{5b^3} - \frac{C_3}{b^7}\right) a^3 + \frac{C_3}{a^4} = 0 \implies C_3 = -\frac{2\beta a^7 b^4}{b^7 - a^7}, B_3 = \frac{2\beta b^4}{5\left(b^7 - a^7\right)}.$$

Lastly, for $\ell = 1$ we obtain

$$B_1 = \frac{\alpha}{a} - \frac{C_1}{a^3},$$
$$B_1 b + \frac{C_1}{b^2} = \left(\frac{\alpha}{a} - \frac{C_1}{a^3}\right) b + \frac{C_1}{b^2} = \frac{3\beta}{5} \implies C_1 = \frac{b^2 a^2 \left(3\beta a - 5\alpha b\right)}{5\left(a^3 - b^2\right)}, B_1 = \frac{1}{a}\left[\alpha - \frac{b^2\left(3\beta a - 5\alpha b\right)}{5\left(a^3 - b^2\right)}\right].$$

In total

$$\Phi(r,\theta) = \begin{cases} \frac{\alpha}{a} r \cos\theta & r < a \\ \left[\frac{1}{a}\left(\alpha - \frac{b^2(3\beta a - 5\alpha b)}{5(a^3-b^2)}\right) r + \frac{1}{r^2}\frac{b^2 a^2(3\beta a - 5\alpha b)}{5(a^3-b^2)}\right]\cos\theta & a < r < b \\ + \left[\frac{2\beta b^4}{5(b^7-a^7)} r^3 - \frac{1}{r^4}\frac{2\beta a^7 b^4}{b^7-a^7}\right]\frac{1}{2}\left(5\cos^3\theta - 3\cos\theta\right) & \\ \frac{3}{5}\frac{\beta b^2}{r^2}\cos\theta + \frac{1}{5}\frac{\beta b^4}{r^4}\left(5\cos^3\theta - 3\cos\theta\right) & r > b \end{cases}$$

Problem 3

Prove that the Green function of a spherical shell $r = a$ is

$$G\left(\vec{r}, \vec{r}'\right) = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \frac{1}{2\ell+1} \left[\frac{r_<^{\ell}}{r_>^{\ell+1}} - \frac{1}{a}\left(\frac{a^2}{rr'}\right)^{\ell+1}\right] Y_{\ell m}^{*}\left(\theta', \varphi'\right) Y_{\ell m}\left(\theta, \varphi\right),$$

where

$$r_> = \max\left(r', r\right), \quad r_> = \min\left(r', r\right).$$

Hint: Use the result you proved in HW3,

$$\frac{1}{\left|\vec{r} - \vec{r}'\right|} = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \frac{1}{2\ell+1} \frac{r_<^{\ell}}{r_>^{\ell+1}} Y_{\ell m}^{*}\left(\theta', \varphi'\right) Y_{\ell m}\left(\theta, \varphi\right).$$

Solution

Recall (HW2) that a conducting sphere admits the Green's function solution

$$G\left(\vec{r}, \vec{r}'\right) = \frac{1}{\left|\vec{r} - \vec{r}'\right|} - \frac{a}{r'\left|\vec{r} - \left(\frac{a}{r'}\right)^2 \vec{r}'\right|},$$

where $\vec{r}$ is the reference point and $\vec{r}'$ is the location of some charge $q = +1$, both outside the sphere. Hence, we just need to find the second term. We define the vectors $\vec{x} = a\hat{r}'$,$\hat{y} = r'r\hat{r}/a$, and obtain

$$-\frac{a}{r'\left|\vec{r} - \left(\frac{a}{r'}\right)^2 \vec{r}'\right|} = -\frac{1}{\left|\frac{r'r}{a}\hat{r} - a\hat{r}'\right|} \equiv -\frac{1}{\left|\vec{y} - \vec{x}\right|}.$$

Since $r', r > a$ we have $y > x$, and we can use the expansion for these vectors to obtain

$$\begin{aligned} -\frac{a}{r'\left|\vec{r} - \left(\frac{a}{r'}\right)^2 \vec{r}'\right|} &= \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \frac{1}{2\ell+1} \frac{x^{\ell}}{y^{\ell+1}} Y_{\ell m}^{*}\left(\theta', \varphi'\right) Y_{\ell m}\left(\theta, \varphi\right) \\ &= \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \frac{1}{2\ell+1} \frac{a^{\ell+1}}{a} \left(\frac{a}{rr'}\right)^{\ell+1} Y_{\ell m}^{*}\left(\theta', \varphi'\right) Y_{\ell m}\left(\theta, \varphi\right) \\ &= \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \frac{1}{2\ell+1} \frac{1}{a} \left(\frac{a^2}{rr'}\right)^{\ell+1} Y_{\ell m}^{*}\left(\theta', \varphi'\right) Y_{\ell m}\left(\theta, \varphi\right), \end{aligned}$$

which when added to the expansion of $1/\left|\vec{r} - \vec{r}'\right|$ gives the desired result

$$G\left(\vec{r}, \vec{r}'\right) = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \frac{1}{2\ell+1} \left[\frac{r_<^{\ell}}{r_>^{\ell+1}} - \frac{1}{a}\left(\frac{a^2}{rr'}\right)^{\ell+1}\right] Y_{\ell m}^{*}\left(\theta', \varphi'\right) Y_{\ell m}\left(\theta, \varphi\right).$$