Homework 4

Question 1

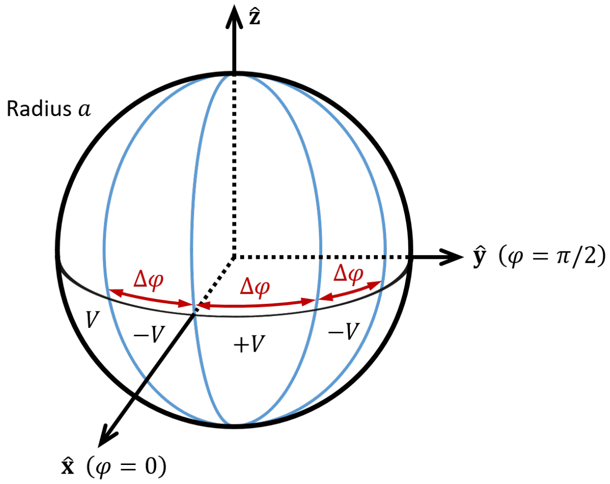
Find the potential outside a sphere of radius R, which has the boundary conditions $\Phi = \phi_1$ on one half and $\Phi = \phi_2$ on the other half, with $V_{1,2}$ constants.

Question 2

Find the electric potential inside of a cylinder of radius a (coaxial with the $\hat{z}$ axis) and height h, where the bases $(z = 0, h)$ are grounded and the potential on the shell is given by $\Phi = V$ for $0 < \varphi < \pi$ and $\Phi = -V$ for $\pi < \varphi < 2\pi$.

Question 3

A spherical shell with radius a is divided to an even number of segments, $2n$, by a set of planes; their common line of intersection is the $\hat{z}$ axis and they are distributed uniformly in the angle φ (see figure). The segments are held at fixed potentials $\pm V$, alternately.



1. Write the potential inside the shell as an expansion in spherical coordinates, and write the integral expression for the coefficients.

2. Show that the coefficients of $Y_{\ell m}$ vanish unless $\ell + m$ is even. **Hint**: Think about the symmetry $z \to -z$ of the setup, and the property of P_ℓ^m under $\cos\theta \to -\cos\theta$.

3. Show that the setup has a symmetry of the form

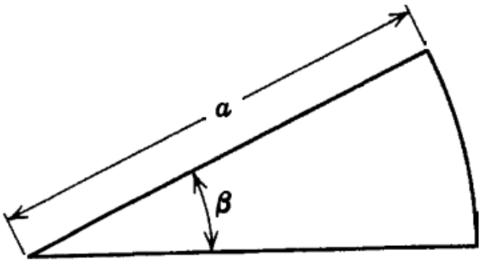
$$\Phi(\varphi) \to A\Phi(\varphi + \Delta\varphi),$$

and find the constant A.

4. Determine for which values of m the coefficient of $Y_{\ell m}$ in the expansion vanish, and write the ones that do not vanish as an integral over $\cos\theta$.

Question 4

A two-dimensional "pizza slice" geometry is defined in polar coordinates by the surfaces $\varphi = 0, \varphi = \beta$ and $\rho = a$, as indicated in the sketch.



Use separation of variables in polar coordinates to show that the Dirichlet Green's function inside the slice ($0 < \varphi < \beta$, $0 < \rho < a$) can be written as

$$G\left(\rho, \varphi; \rho', \varphi'\right) = \sum_{m=1}^{\infty} \frac{4}{m} \rho_<^{m\pi/\beta} \left(\frac{1}{\rho_>^{m\pi/\beta}} - \left(\frac{\rho_>}{a^2}\right)^{m\pi/\beta} \right) \sin\left(\frac{m\pi\varphi}{\beta}\right) \sin\left(\frac{m\pi\varphi'}{\beta}\right).$$

Guidance: Recall the method used in question 4 of HW3, and use the completeness relation

$$\delta\left(\varphi - \varphi'\right) = \frac{2}{\beta} \sum_{m=1}^{\infty} \sin\left(\frac{m\pi\varphi}{\beta}\right) \sin\left(\frac{m\pi\varphi'}{\beta}\right).$$

Separating the solution to a radial function $g_m\left(\rho, \rho'\right)$ and a suitable angular part, choose appropriate boundary conditions. Show that g_m is symmetric under $\rho \leftrightarrow \rho'$, and use that to prove that the radial solution must be

$$g_m\left(\rho, \rho'\right) \propto \rho_<^{m\pi/\beta} \left(\frac{1}{\rho_>^{m\pi/\beta}} - \left(\frac{\rho_>}{a^2}\right)^{m\pi/\beta} \right).$$