Homework 4

Question 1

Find the potential outside a sphere of radius R, which has the boundary conditions $\Phi = \phi_1$ on one half and $\Phi = \phi_2$ on the other half, with $V_{1,2}$ constants.

Solution

The general solution to Laplace's equation in spherical coordinates in the case of azimuthal independence is given by

$$\phi(r,\theta) = \sum_{l=0}^{\infty} \left(A_l r^l + \frac{B_l}{r^{l+1}} \right) P_l(\cos\theta)$$

Outside the sphere, $r > R$ the potential must be finite, so $A_l = 0$.
On the surface of the sphere $r = R$,

$$\phi(R,\theta) = \sum_{l=0}^{\infty} \left(\frac{B_l}{R^{l+1}} \right) P_l(\cos\theta) = V(\theta)$$

where $V(0 \le \theta \le \pi/2) = \phi_1$ and $V(\pi/2 \le \theta \le \pi) = \phi_2$.

$$\int_{-1}^{1} \sum_{l=0}^{\infty} \left(\frac{B_l}{R^{l+1}} \right) P_l(\cos\theta) P_{l'}(\cos\theta) d(\cos\theta) = \int_{-1}^{1} V(\theta) P_{l'}(\cos\theta) d(\cos\theta)$$

Using the orthogonality relation

$$\int_{-1}^{1} P_{l'}(\mu) P_{l'}(\mu) d(\mu) = \frac{2}{2l'+1} \delta_{l,l'}$$

We get

$$\frac{B_l}{R^{l+1}} \frac{2}{2l+1} = \int_{-1}^{1} V(\theta) P_{l'}(\cos\theta) d(\cos\theta) = \phi_1 \int_{-1}^{0} P_l(\mu) d(\mu) + \phi_2 \int_{0}^{1} P_l(\mu) d(\mu)$$

Or,

$$B_l = R^{l+1} \frac{2l+1}{2} \left[\phi_1 \int_{-1}^{0} P_l(\mu) d(\mu) + \phi_2 \int_{0}^{1} P_l(\mu) d(\mu) \right]$$

Since Legendre polynomials satisfy $P_l(\mu) = (-1)^l P_l(-\mu)$,

$$B_l = R^{l+1} \frac{2l+1}{2} I_l \begin{cases} \phi_1 + \phi_2 & , l = 2m \\ \phi_2 - \phi_1 & , l = 2m+1 \end{cases}$$

and $I_l = \int_0^1 P_l(\mu) d(\mu)$, this integral may be evaluated by integrating both sides of the legendre equation:

$$\int_{0}^{1} \left[\frac{d}{d\mu} \left[(1-\mu^2) \frac{dP_l(\mu)}{d\mu} \right] + l(l+1) P_l(\mu) \right] d\mu = 0$$

$$(1-\mu^2)\frac{dP_l}{d\mu}|_0^1 + l(l+1)\int_0^1 P_l(\mu)d\mu = 0$$

So we obtain

$$I_l = \frac{P_l'(0)}{l(l+1)}$$

Notice that if P_l is an even function, P_l' is an odd function, so $P_l'(0) = 0$ and finally $B_l = 0$ for even l.

$$B_l = \begin{cases} 0 & ,l = 2m \\ \frac{2l+1}{2l(l+1)} P_l'(0) R^{l+1}(\phi_2 - \phi_1) & ,l = 2m+1 \end{cases}$$

The first few legendres are $P_1 = \mu, P_3 = \frac{1}{2}(5\mu^3 - 3\mu)$. The first two terms in the solution are then

$$\phi(r,\theta) = \frac{3}{2}\left(\frac{R}{r}\right)^2 \frac{\phi_2 - \phi_1}{2} P_1(\cos\theta) - \frac{7}{8}\left(\frac{R}{r}\right)^4 \frac{\phi_2 - \phi_1}{2} P_3(\cos\theta) + ...$$

Question 2

Find the electric potential inside of a cylinder of radius a (coaxial with the $\hat{z}$ axis) and height h, where the bases $(z = 0, h)$ are grounded and the potential on the shell is given by $\Phi = V$ for $0 < \varphi < \pi$ and $\Phi = -V$ for $\pi < \varphi < 2\pi$.

Solution

$$\frac{1}{r}\frac{\partial}{\partial r}r\frac{\partial\varphi}{\partial r}+\frac{1}{r^2}\frac{\partial^2\varphi}{\partial\phi^2}+\frac{\partial^2\varphi}{\partial z^2}=0. \tag{1}$$

Boundary conditions:

$$\varphi(z=0)=\varphi(z=h)=0;\quad \varphi(\phi)=\varphi(\phi+2\phi);\quad \varphi(r=a)=\begin{cases}V; & 0<\phi<\pi;\\ -V; & \pi<\phi<2\pi.\end{cases} \tag{2}$$

Separating variables, one looks solutions in the form $\varphi(r,\phi,z)=R(r)Z(z)\Phi(\phi)$, where the functions R, Z, and Φ satisfy equations

$$\frac{d^2Z}{dz^2}=-k^2Z; \tag{3}$$

$$\frac{d^2R}{dr^2}+\frac{1}{r}\frac{dR}{dr}-\left(k^2+\frac{m^2}{r^2}\right)R=0; \tag{4}$$

$$\frac{d^2\Phi}{d\phi^2}=-m^2\Phi. \tag{5}$$

One has to choose m to be integer so that the solution is 2π-periodic in ϕ. The homogeneous boundary conditions at $z=0$ and h are satisfied by choosing $k=\pi l/h$, where l is integer. The non-singular solution to eq. (4) is the modified Bessel function $R=I_m(kr)$.

Now the solution is written as

$$\varphi(r,\phi,z)=\sum_{m,l}\left[C_{ml}\sin m\phi+D_{ml}\cos m\phi\right]\sin\frac{\pi lz}{h}I_m\left(\frac{\pi lr}{h}\right). \tag{6}$$

The constants are found from the boundary conditions at $r=a$. Multiplying eq. (6) by $\sin m\phi\sin(\pi lz/h)$ and making use of the orthogonality conditions, $\int_0^\pi \sin mx\sin m'x dx=\frac{\pi}{2}\delta_{mm'}$ and $\int_0^{2\pi}\sin mx\sin m'xdx=\pi\delta_{mm'}$, one gets

$$V\left(\int_0^\pi \sin m\phi d\phi-\int_\pi^{2\pi}\sin m\phi d\phi\right)\int_0^h \sin\left(\frac{\pi lz}{h}\right)dz=\frac{\pi h}{2}C_{ml}I_m\left(\frac{\pi la}{h}\right). \tag{7}$$

The lhs vanishes for even m and l, and so does C_{ml}. For odd m and l, one gets

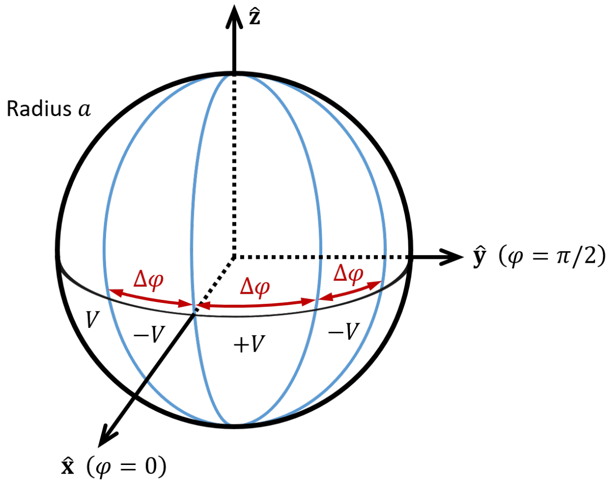
$$C_{ml}=\frac{16V}{\pi^2 mlI_l\left(\frac{l\pi a}{h}\right)};\quad m=2s+1;\quad l=2n+1;\quad s,\,n=0,\,1,\,2\,\ldots.$$

The similar calculation shows that for any m and l, $D_{ml}=0$. Now one can finally present the solution in the form

$$\varphi(r,\phi,z)=\frac{16}{\pi^2}V\sum_{n,s}\frac{\sin\frac{(2n+1)\pi z}{h}\sin(2s+1)\phi}{(2s+1)(2n+1)I_{2n+1}\left(\frac{(2n+1)\pi a}{h}\right)}I_{2n+1}\left(\frac{(2n+1)\pi r}{h}\right). \tag{8}$$

Question 3

A spherical shell with radius a is divided to an even number of segments, $2n$, by a set of planes; their common line of intersection is the $\hat{z}$ axis and they are distributed uniformly in the angle φ (see figure). The segments are held at fixed potentials $\pm V$, alternately.



1. Write the potential inside the shell as an expansion in spherical coordinates, and write the integral expression for the coefficients.

2. Show that the coefficients of $Y_{\ell m}$ vanish unless $\ell + m$ is even. **Hint**: Think about the symmetry $z \to -z$ of the setup, and the property of P_ℓ^m under $\cos\theta \to -\cos\theta$.

3. Show that the setup has a symmetry of the form

$$\Phi(\varphi) \to A\Phi(\varphi + \Delta\varphi),$$

   and find the constant A.

4. Determine for which values of m the coefficient of $Y_{\ell m}$ in the expansion vanish, and write the ones that do not vanish as an integral over $\cos\theta$.

Solution

1. The solution to the Laplace equation (without azimuthal symmetry!) is

$$\Phi(r, \theta, \varphi) = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \left[A_{\ell,m} r^\ell + B_{\ell,m} r^{-\ell-1}\right] Y_{\ell,m}(\theta, \varphi),$$

and since we want to avoid a divergence at $r=0$ we set $B_{\ell,m}=0$ and thus

$$\Phi(r,\theta,\varphi)=\sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell}A_{\ell,m}r^{\ell}Y_{\ell,m}(\theta,\varphi).$$

We find the coefficients by equating across $r=a$,

$$\Phi(r=a,\theta,\varphi)=\sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell}A_{\ell,m}a^{\ell}Y_{\ell,m}(\theta,\varphi),$$

where the boundary condition is

$$\Phi_0(\varphi)=\begin{cases}+V & ,\text{if } \frac{2\pi}{2n}j<\varphi<\frac{2\pi}{2n}(j+1) \text{ for even } j\\ -V & ,\text{if } \frac{2\pi}{2n}j<\varphi<\frac{2\pi}{2n}(j+1) \text{ for odd } j\end{cases}.$$

We use the orthogonality of $Y_{\ell m}$ and multiply by $\int Y^{*}_{\ell' m'}(\theta,\varphi)\sin\theta\,\mathrm{d}\Omega$. We obtain the coefficients in integral form,

$$A_{\ell,m}=\int_0^{2\pi}\mathrm{d}\varphi\int_0^{\pi}\sin\theta\,\mathrm{d}\theta\phi_0(\varphi)Y^{*}_{\ell,m}(\theta,\varphi)=\int\mathrm{d}\Omega\phi_0(\varphi)Y^{*}_{\ell,m}(\theta,\varphi).$$

2. We express the parity property $z\to -z$ in spherical coordinates as $\cos\theta\to-\cos\theta$. Therefore,

$$\begin{aligned}Y_{\ell,m}(\theta,\varphi)&=\sqrt{\frac{2\ell+1}{4\pi}\frac{(\ell-m)!}{(\ell+m)!}}P_{\ell}^{m}(\cos\theta)e^{im\varphi}\\&=(-)^{\ell+m}\sqrt{\frac{2\ell+1}{4\pi}\frac{(\ell-m)!}{(\ell+m)!}}P_{\ell}^{m}(-\cos\theta)e^{im\varphi}\\&=(-)^{\ell+m}Y_{\ell,m}(\pi-\theta,\varphi)\end{aligned}$$

where we used $P_{\ell}(-x)=(-)^{\ell}P_{\ell}(x)$, $-\cos\theta=\cos(\pi-\theta)$ and

$$P_{\ell}^{m}=(-1)^{m}\left(1-x^2\right)^{m/2}\mathrm{d}^{m}P_{\ell}(x)/\mathrm{d}x^{m}.$$

Therefore we see that the only terms in the expansion which satisfy the symmetry of the system (i.e. $Y_{\ell m}(\theta,\varphi)=Y_{\ell m}(\pi-\theta,\varphi)$) are the ones with $\ell+m$ even.

3. The boundary condition dictates that

$$\Phi_0(\varphi) = -\Phi_0(\varphi + \frac{2\pi}{2n}),$$

and thus $A = -1$. Since the Laplace equation is invariant under this transformation (i.e if $\Phi(r,\theta,\varphi)$ is a solution, so is $\tilde{\Phi}(r,\theta,\varphi) = -\Phi(r,\theta,\varphi+\Delta\varphi)$), from uniqueness of the solution we obtain

$$\Phi(\varphi) = -\Phi(\varphi + \frac{2\pi}{2n}).$$

4. We compute directly,

$$A_{\ell,m} \propto \int_0^{2\pi} \mathrm{d}\varphi\, \Phi_0(\varphi) e^{-im\varphi} = \sum_{j=0}^{2n-1} \int_{\pi j/n}^{\pi(j+1)/n} \mathrm{d}\varphi\, (-)^j V e^{-im\varphi} = \tilde{A}_m.$$

For $m = 0$ we find

$$\tilde{A}_0 = \sum_{j=0}^{2n-1} \int_{\pi j/n}^{\pi(j+1)/n} \mathrm{d}\varphi\, (-)^j V = \frac{\pi V}{n} \sum_{j=0}^{2n-1} (-)^j = 0,$$

and for $m \neq 0$ we have

$$\begin{aligned}
\tilde{A}_m &= \sum_{j=0}^{2n-1} (-)^j V \left[\frac{e^{-im\varphi}}{-im}\right]_{\varphi=\pi j/n}^{\pi(j+1)/n} \\
&= -\frac{V}{im} \sum_{j=0}^{2n-1} (-)^j e^{-i\pi m j/n} \left(e^{-i\pi m/n} - 1\right) \\
&= -\frac{V}{im} \sum_{j=0}^{2n-1} (-e^{-i\pi m/n})^j \left(e^{-i\pi m/n} - 1\right).
\end{aligned}$$

This is the sum of a geometric series: For $-e^{-i\pi m/n} \neq 1$ we find

$$\tilde{A}_m = -\frac{V}{im} \frac{(-e^{-i\pi m/n})^{2n} - 1}{-e^{-i\pi m/n} - 1} \left(e^{-i\pi m/n} - 1\right) = 0,$$

and for $-e^{-i\pi m/n} = 1$,

$$\tilde{A}_m = \frac{2V}{im} \sum_{j=0}^{2n-1} 1 = \frac{4nV}{im},$$

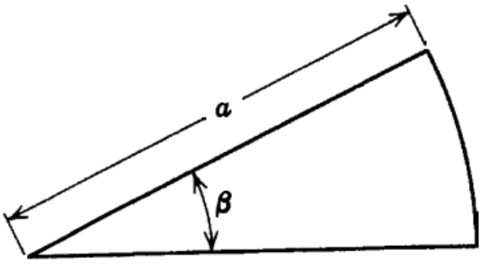
meaning that $A_{\ell,m} = 0$ unless $m = (2k-1)\,n$ for $k \in \mathbb{Z}$. Therefore, the full integral for the coefficients is

$$A_{\ell,(2k-1)n} = \sqrt{\frac{2\ell+1}{4\pi}\frac{[\ell-(2k-1)n]!}{[\ell+(2k-1)n]!}} \int_{-1}^{1} \mathrm{d}(\cos\theta) P_\ell^{(2k-1)n}(\cos\theta) \int_{0}^{2\pi} \mathrm{d}\varphi\, \Phi_0(\varphi) e^{-i(2k-1)n\varphi}$$

$$= \frac{4nV}{im}\sqrt{\frac{2\ell+1}{4\pi}\frac{[\ell-(2k-1)n]!}{[\ell+(2k-1)n]!}} \int_{-1}^{1} \mathrm{d}(\cos\theta) P_\ell^{(2k-1)n}(\cos\theta).$$

Question 4

A two-dimensional "pizza slice" geometry is defined in polar coordinates by the surfaces $\varphi = 0, \varphi = \beta$ and $\rho = a$, as indicated in the sketch.



Use separation of variables in polar coordinates to show that the Dirichlet Green's function inside the slice ($0 < \varphi < \beta$, $0 < \rho < a$) can be written as

$$G(\rho, \varphi; \rho', \varphi') = \sum_{m=1}^{\infty} \frac{4}{m} \rho_<^{m\pi/\beta} \left(\frac{1}{\rho_>^{m\pi/\beta}} - \left(\frac{\rho_>}{a^2} \right)^{m\pi/\beta} \right) \sin\left(\frac{m\pi\varphi}{\beta} \right) \sin\left(\frac{m\pi\varphi'}{\beta} \right).$$

Guidance: Recall the method used in question 4 of HW3, and use the completeness relation

$$\delta(\varphi - \varphi') = \frac{2}{\beta} \sum_{m=1}^{\infty} \sin\left(\frac{m\pi\varphi}{\beta} \right) \sin\left(\frac{m\pi\varphi'}{\beta} \right).$$

Separating the solution to a radial function $g_m(\rho, \rho')$ and a suitable angular part, choose appropriate boundary conditions. Show that g_m is symmetric under $\rho \leftrightarrow \rho'$, and use that to prove that the radial solution must be

$$g_m(\rho, \rho') \propto \rho_<^{m\pi/\beta} \left(\frac{1}{\rho_>^{m\pi/\beta}} - \left(\frac{\rho_>}{a^2} \right)^{m\pi/\beta} \right).$$

Solution

The Green function solves the Poisson equation

$$\nabla^2 G = -4\pi\delta^{(3)}(\vec{r} - \vec{r}') = -\frac{4\pi}{\rho} \delta(\rho - \rho') \delta(\varphi - \varphi'),$$

where we have seen many times that the basis of sines satisfies the completeness relation

$$\delta(\varphi - \varphi') = \frac{2}{\beta} \sum_{m=1}^{\infty} \sin\left(\frac{m\pi\varphi}{\beta} \right) \sin\left(\frac{m\pi\varphi'}{\beta} \right).$$

Therefore,

$$\nabla^2 G = -\frac{8\pi}{\beta\rho}\delta\left(\rho-\rho'\right)\sum_{m=1}^{\infty}\sin\left(\frac{m\pi\varphi}{\beta}\right)\sin\left(\frac{m\pi\varphi'}{\beta}\right),$$

and we can expand the Green function in terms of $\sin\left(m\pi\varphi/\beta\right)\sin\left(m\pi\varphi'/\beta\right)$:

$$G\left(\rho,\varphi;\rho',\varphi'\right) = \sum_{m=1}^{\infty} g_m\left(\rho,\rho'\right)\sin\left(\frac{m\pi\varphi}{\beta}\right)\sin\left(\frac{m\pi\varphi'}{\beta}\right).$$

Plugging this form into the polar Poisson equation, we find for the radial function

$$\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial g_m}{\partial\rho}\right) - \frac{g_m}{\rho^2}\left(\frac{m\pi}{\beta}\right)^2 = -\frac{8\pi}{\beta\rho}\delta\left(\rho-\rho'\right). \tag{1}$$

We separate the problem to $\rho=\rho'$ and $\rho\neq\rho'$: For $\rho\neq\rho'$ we have the simpler equation

$$\frac{\partial}{\partial\rho}\left(\rho\frac{\partial g_m}{\partial\rho}\right) = \frac{g_m}{\rho}\left(\frac{m\pi}{\beta}\right)^2,$$

for which the general solution (as you've seen in class) is

$$g_m\left(\rho,\rho'\right) = A_m\left(\rho'\right)\rho^{m\pi/\beta} + B_m\left(\rho'\right)\rho^{-m\pi/\beta}.$$

For a Dirichlet problem, we choose the boundary conditions to be vanishing on all the boundaries ($\rho=a$ and $\varphi=0,\beta$). Therefore,

$$g_m\left(\rho=a;\rho'\right)=0.$$

We wish to keep the radial solution regular at $\rho<\rho'$, and thus we choose

$$g_m\left(\rho<\rho';\rho'\right) = A_m\rho^{m\pi/\beta}.$$

Using the boundary condition $g_m\left(\rho=a;\rho'\right)=0$, for $\rho>\rho'$ we obtain

$$\begin{aligned}
0 &= B_m a^{m\pi/\beta} + C_m a^{-m\pi/\beta}\\
\implies B_m &= -C_m a^{-2m\pi/\beta}\\
\implies g_m\left(\rho,\rho'\right) &= \begin{cases} A_m\left(\rho'\right)\rho^{m\pi/\beta} & \rho<\rho' \\ B_m\left(\rho'\right)\left(\rho^{-m\pi/\beta} - a^{-2m\pi/\beta}\rho^{m\pi/\beta}\right) & \rho>\rho' \end{cases}
\end{aligned}$$

We use continuity at

$$g_m\left(r\to r'^{+}\right) = g_m\left(r\to r'^{-}\right),$$

to obtain

$$A_m(\rho')(\rho')^{m\pi/\beta} = B_m(\rho')\left((\rho')^{-m\pi/\beta} - a^{-2m\pi/\beta}(\rho')^{m\pi/\beta}\right),$$

which gives the solution

$$g_m(\rho,\rho') = \begin{cases} B_m(\rho')\,\rho^{m\pi/\beta}(\rho')^{-m\pi/\beta}\left(\frac{1}{(\rho')^{m\pi/\beta}} - \left(\frac{\rho'}{a^2}\right)^{m\pi/\beta}\right) & \rho<\rho' \\ B_m(\rho')\left(\frac{1}{\rho^{m\pi/\beta}} - \left(\frac{\rho}{a^2}\right)^{m\pi/\beta}\right) & \rho>\rho' \end{cases}$$

We can now use the symmetry $\rho \leftrightarrow \rho'$ to obtain the general solution; Switching the variables, we find the radial function

$$g_m(\rho,\rho') = \begin{cases} B_m(\rho)\,(\rho')^{m\pi/\beta}(\rho)^{-m\pi/\beta}\left(\frac{1}{\rho^{m\pi/\beta}} - \left(\frac{\rho}{a^2}\right)^{m\pi/\beta}\right) & \rho>\rho' \\ B_m(\rho)\left(\frac{1}{(\rho')^{m\pi/\beta}} - \left(\frac{\rho'}{a^2}\right)^{m\pi/\beta}\right) & \rho<\rho' \end{cases}$$

and equating $g_m(\rho,\rho') = g_m(\rho',\rho)$, we obtain

$$\begin{gathered} B_m(\rho')(\rho')^{-m\pi/\beta} = B_m(\rho)\,\rho^{-m\pi/\beta} \\ \implies B_m(\rho) = C_m\rho^{m\pi/\beta}, \end{gathered}$$

with C_m a constant independent of ρ,ρ'. Therefore, the solution in both regions has the form

$$g_m(\rho,\rho') = \begin{cases} C_m\rho^{m\pi/\beta}\left(\frac{1}{(\rho')^{m\pi/\beta}} - \left(\frac{\rho'}{a^2}\right)^{m\pi/\beta}\right) & \rho<\rho' \\ C_m(\rho')^{m\pi/\beta}\left(\frac{1}{\rho^{m\pi/\beta}} - \left(\frac{\rho}{a^2}\right)^{m\pi/\beta}\right) & \rho>\rho' \end{cases}$$

or, using $\rho_< = \min(\rho,\rho')$ and $\rho_> = \max(\rho,\rho')$,

$$g_m(\rho,\rho') = C_m\rho_<^{m\pi/\beta}\left(\frac{1}{\rho_>^{m\pi/\beta}} - \left(\frac{\rho_>}{a^2}\right)^{m\pi/\beta}\right).$$

Finally, we use the discontinuity caused by the delta function,

$$\lim_{\epsilon\to0}\int_{\rho'-\epsilon}^{\rho'+\epsilon}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial g_m}{\partial\rho}\right)\mathrm{d}\rho = \rho\frac{\partial g_m}{\partial\rho}\bigg|_{\rho\to\rho'^+} - \rho\frac{\partial g_m}{\partial\rho}\bigg|_{\rho\to\rho'^-} = -\frac{8\pi}{\beta}\lim_{\epsilon\to0}\int_{\rho'-\epsilon}^{\rho'+\epsilon}\delta(\rho-\rho')\,\mathrm{d}\rho = -\frac{8\pi}{\beta},$$

and obtain the coefficients C_m,

$$-\left(m\pi/\beta\right)C_m\left[1+\left(\frac{\rho'}{a}\right)^{2m\pi/\beta}+1-\left(\frac{\rho'}{a}\right)^{2m\pi/\beta}\right]=-\frac{8\pi}{\beta}$$
$$\implies C_m=\frac{4}{m}.$$

Therefore, the complete solution is

$$G\left(\rho,\varphi;\rho',\varphi'\right)=\sum_{m=1}^{\infty}\frac{4}{m}\rho_<^{m\pi/\beta}\left(\frac{1}{\rho_>^{m\pi/\beta}}-\left(\frac{\rho_>}{a^2}\right)^{m\pi/\beta}\right)\sin\left(\frac{m\pi\varphi}{\beta}\right)\sin\left(\frac{m\pi\varphi'}{\beta}\right),$$

as desired.