Class Exercise 5 - Magnetostatics and dielectrics

In previous classes, we solved the Laplace and Poisson equations for electrostatic systems. In this class, we finally tackle magnetostatic systems with stationary currents, as well as dielectric materials.

Magnetostatics

As seen in the lectures, the Maxwell equation $\vec{\nabla} \cdot \vec{B} = 0$ implies the magnetic field is the curl of a vector function $\vec{A}$, such that

$$\vec{\nabla} \times \vec{A} = \vec{B},$$
$$\vec{\nabla} \times \left(\vec{\nabla} \times \vec{A}\right) = \frac{4\pi}{c} \vec{J},$$

where $\vec{J}$ is the current. The vector potential $\vec{A}$ is not unique unless a gauge is specified. In Coulomb gauge,

$$\vec{\nabla} \cdot \vec{A} = 0,$$

and therefore $\vec{A}$ solves the Maxwell equation

$$\nabla^2 \vec{A} = -\frac{4\pi}{c} \vec{J},$$

which is a Poisson equation per each component A^i. The general solution in infinite space is then

$$\vec{A}\left(\vec{r}\right) = \frac{1}{c} \iiint \frac{\vec{J}\left(\vec{r}'\right) \mathrm{d}^3 \vec{r}'}{\left|\vec{r} - \vec{r}'\right|}.$$

The boundary conditions across some surface are gotten by integrating the Maxwell equation for the magnetic field, and generically come in the form

$$B_{\parallel}^{+} - B_{\parallel}^{-} = \frac{4\pi}{c} \vec{J},$$
$$B_{\perp}^{+} = B_{\perp}^{-}.$$

Problem 1

Given the current density per unit length

$$\vec{j} = j_0 \left(\sin \left(k_0 y \right) \hat{x} + \sin \left(k_0 x \right) \hat{y} \right),$$

in the $x - y$ plane, find the magnetic field.

Solution

We were given a current density, which calls for the use of the Maxwell equation

$$\nabla \times B = \frac{4\pi}{c} \vec{J}.$$

In terms of the vector potential, In Coulomb gauge we need to solve

$$\nabla^2 \vec{A} = -\frac{4\pi}{c} \vec{J}.$$

We have two separate components for the $\hat{x}$ and $\hat{y}$ axes in this equation. Using superposition, we may separate the problem to vector potentials A_x and A_y with sources j_x and j_y, respectively. When we are done, adding the two solutions will give the complete solution.

In $\hat{x}$ we find

$$\nabla^2 A_x = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \right) A_x \left(x, y, z \right) = \frac{4\pi}{c} j_0 \sin \left(k_0 y \right) \delta \left(z \right).$$

The A_x component is independent of x, since from the gauge condition,

$$\vec{\nabla} \cdot A_x \hat{x} = \frac{\partial}{\partial x} A_x = 0.$$

Anywhere $z \neq 0$, we have the Laplace equation

$$\left(\frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \right) A_x \left(y, z \right) = 0,$$

and thus we choose a separable solution

$$A_x \left(y, z \right) = Y \left(y \right) Z \left(z \right).$$

Therefore, we have

$$\frac{1}{Y} \frac{d^2 Y}{dy^2} + \frac{1}{Z} \frac{d^2 Z}{dz^2} = 0,$$

with the boundary condition $A_x(z \to \pm\infty) = 0$, since the current at $z = 0$ has the fluctuating form of a sine, which in average (i.e. in the far field multipole expansion) is vanishing. Thus, we choose a constant k^2 such that

$$\frac{d^2 Z}{dz^2} = k^2 Z \implies Z \propto e^{\pm kz},$$
$$\frac{d^2 Y}{dy^2} = -k^2 Y \implies Y \propto \sin(ky + \varphi).$$

To satisfy the boundary condition at $z \to \pm\infty$, we must have

$$A_x(y,z) = \begin{cases} A\sin(ky+\varphi)\, e^{-kz} & z > 0 \\ B\sin(ky+\varphi)\, e^{kz} & z < 0 \end{cases}$$

The magnetic field component are then

$$B_x = 0,\ B_y = \left(\vec{\nabla} \times A_x \hat{x}\right) \cdot \hat{y} = \frac{\partial A_x}{\partial z},\ B_z = \left(\vec{\nabla} \times A_x \hat{x}\right) \cdot \hat{z} = -\frac{\partial A_x}{\partial y},$$

so

$$B_y(y,z) = \begin{cases} -kA\sin(ky+\varphi)\, e^{-kz} & z > 0 \\ kB\sin(ky+\varphi)\, e^{kz} & z < 0 \end{cases}$$

We can use the delta-function boundary condition at $z = 0$ to obtain the jump condition you saw in class,

$$B_{\parallel}\Big|_{z=0^+} - B_{\parallel}\Big|_{z=0^-} = \frac{4\pi}{c}\, \vec{J}\Big|_{z=0} \times \hat{n},$$

where here the parallel component under consideration is $\hat{y}$, and the normal to the surface is $\hat{n} = \hat{z}$. We use the Maxwell equation with the source to obtain

$$\oiint \left(\vec{\nabla} \times \vec{B}\right) \cdot \mathrm{d}\vec{S} = \int \vec{B} \cdot \mathrm{d}\vec{\ell} = -\frac{4\pi}{c} \oiint J_x \hat{x} \cdot \mathrm{d}\vec{S} = -\frac{4\pi}{c} j_0 \int \sin(k_0 y)\, \mathrm{d}y \int \delta(z)\, \mathrm{d}z,$$

where $j_x = j_0 \sin(ky)\, \delta(z)$ and S is the infinitesimal surface in figure 1. We take the sides $(\mathrm{d}z, \mathrm{d}x)$ of the square to zero, and thus only the components parallel to the top and the bottom where $d\vec{\ell} = \pm\hat{y}$ enter the condition. For and integral between $y, y + \delta y$ for a sufficiently small δy for which $B_y(y+\delta y) \approx B_y(y)$, $j_x(y+\delta y) = j_x(y)$, we find that

$$\int \vec{B} \cdot \mathrm{d}\vec{\ell} = \left(B_y\left(y, z = 0^+\right) - B_y\left(y, z = 0^-\right)\right) \delta y = -\frac{4\pi}{c} j_0 \delta y \sin(k_0 y).$$

We find that

$$B_y\left(y, z=0^+\right) - B_y\left(y, z=0^-\right) = -\frac{4\pi}{c} j_0 \sin\left(ky\right)$$
$$\Longrightarrow\ -kA\sin\left(ky+\varphi\right) - kB\sin\left(ky+\varphi\right) = -\frac{4\pi}{c} j_0 \sin\left(ky\right).$$

Hence, $\varphi = 0$, $k = k_0$ and $k_0\left(A+B\right) = 4\pi j_0/c$. We can use this result to find B_z,

$$B_z = -\frac{\partial A_x}{\partial y} = \begin{cases} -kA\cos\left(ky+\varphi\right)e^{-kz} & z>0 \\ -kB\cos\left(ky+\varphi\right)e^{kz} & z<0 \end{cases}$$
$$= -k_0 \begin{cases} A\cos\left(k_0 y\right)e^{-k_0 z} & z>0 \\ B\cos\left(k_0 y\right)e^{k_0 z} & z<0 \end{cases}$$

As we found before, the field is continuous in the $\hat{z}$ direction, which indeed agrees with the boundary condition

$$B_\perp\big|_{z=0^+} = B_\perp\big|_{z=0^-}.$$

We can derive this condition directly, of course. Using the surface in figure 1, we have

$$\iiint \vec{\nabla}\cdot\vec{B}\,\mathrm{d}V = \oint\!\!\oint_{\partial V} \vec{B}\cdot\mathrm{d}\vec{S} = 0.$$

The sides of the square are taken to zero, so the only components taken into consideration are in the $\hat{z}$ direction. We have

$$\oint\!\!\oint_{\partial V} \vec{B}\cdot\mathrm{d}\vec{S} = \int \left[B_z\left(y, z=0^+\right)\hat{z}\cdot\hat{z} - B_z\left(y, z=0^-\right)\hat{z}\cdot\hat{z}\right]\mathrm{d}y = 0.$$

We take the top and bottom of the surface to be of infinitesimal length δy, such that B_z does not vary much between y and $y+\delta y$. Therefore, we get the continuity of the perpendicular magnetic field,

$$B_z\left(y, z=0^+\right) = B_z\left(y, z=0^-\right).$$

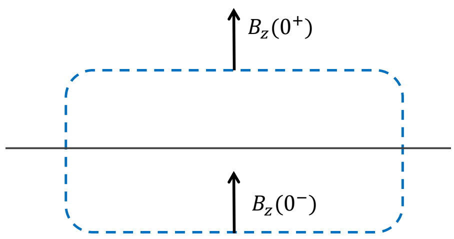

Fig. 1: Infinitesimal surface surrounding the plane $z=0$.

From the continuity of B_z across $z = 0$ we find $A = B$, and along with the jump condition, we have $k_0 A = 2\pi j_0/c$. The magnetic field obtained from $\vec{A}_1 = A_x \hat{x}$ is then

$$\vec{B}_1 = \frac{2\pi j_0}{c} \begin{cases} \left(-\sin\left(k_0 y\right)\hat{y} - \cos\left(k_0 y\right)\hat{z}\right) e^{-k_0 z} & z > 0 \\ \left(\sin\left(k_0 y\right)\hat{y} - \cos\left(k_0 y\right)\hat{z}\right) e^{k_0 z} & z < 0 \end{cases}$$

Similarly, repeating the entire analysis for $\vec{A}_2 = A_y \hat{y}$ results in the magnetic field

$$\vec{B}_2 = \frac{2\pi j_0}{c} \begin{cases} \left(\sin\left(k_0 x\right)\hat{x} - \cos\left(k_0 y\right)\hat{z}\right) e^{-k_0 z} & z > 0 \\ \left(-\sin\left(k_0 x\right)\hat{x} - \cos\left(k_0 y\right)\hat{z}\right) e^{k_0 z} & z < 0 \end{cases}$$

Note that due to the symmetry in z, $B_z^{(1)} = B_z^{(2)}$. We also have $B_x^{(2)} = -B_y^{(1)}\,(y = x, z)$, which follows from the flipping of the sign of the cross product in the jump condition, where now

$$B_x\left(z = 0^+\right)\hat{x} - B_x\left(z = 0^-\right)\hat{x} = -\frac{4\pi}{c}\int_{z=0^-}^{z=0^+} dz\hat{z} \times \hat{y} j_y = +\frac{4\pi}{c} j_0 \sin\left(k_0 x\right)\hat{x}.$$

Problem 2

Find the magnetic field of a cylindrical shell of radius R, with a current density (per unit length) parallel to the cylinder's axis

$$j = \begin{cases} I_0 & 0 < \varphi < \pi \\ -I_0 & \pi < \varphi < 2\pi \end{cases}$$

Solution

Due to the symmetry in the $\hat{z}$ axis, $\vec{A} = \vec{A}\left(r, \phi\right)$ only. In infinite space,

$$A^i\left(\vec{r}\right) = \int \frac{J^i\left(\vec{r}'\right) \mathrm{d}^3\vec{r}'}{c\left|\vec{r} - \vec{r}'\right|},$$

so we already see that $\vec{A}$ should align with $\vec{J}$ and only posses a $\hat{z}$ component.

We have

$$\nabla^2 A_z\left(r, \phi\right) = -\frac{4\pi}{c}\delta\left(r - R\right)\begin{cases} I_0 & 0 < \phi < \pi \\ -I_0 & \pi < \phi < 2\pi \end{cases},$$

which for $r \neq R$ is a Laplace equation which we solve in cylindrical coordinates. We therefore write the general expression

$$A_z\left(r, \phi\right) = \begin{cases} \sum_{m=0}^{\infty} r^m\left(a_m \cos\left(m\phi\right) + b_m \sin\left(m\phi\right)\right) & r < R \\ d_0 \ln r + \sum_{m=0}^{\infty} \frac{1}{r^m}\left(e_m \cos\left(m\phi\right) + f_m \sin\left(m\phi\right)\right) & r > R \end{cases}$$

Across the boundary at $r = R$, B_ϕ jumps while B_r is continuous. Therefore, for

$$B_r = \left(\vec{\nabla} \times A_z\hat{z}\right)_r = \frac{1}{r}\frac{\partial A_z}{\partial \phi},$$

we have the condition

$$\left.\frac{\partial A_z}{\partial \phi}\right|_{r=R^+} = \left.\frac{\partial A_z}{\partial \phi}\right|_{r=R^-},$$

which gives

$$\sum_{m=0}^{\infty} \frac{m}{R^m}\left(-e_m \sin\left(m\phi\right) + f_m \cos\left(m\phi\right)\right) = \sum_{m=0}^{\infty} mR^m\left(-a_m \sin\left(m\phi\right) + b_m \cos\left(m\phi\right)\right)$$

$$\text{orthogonality} \implies e_m = R^{2m}a_m,\ f_m = R^{2m}b_m.$$

The ϕ component is

$$B_\phi = -\frac{\partial A_z}{\partial r},$$

giving the jump condition

$$-\left.\frac{\partial A_z}{\partial r}\right|_{r=R^+} + \left.\frac{\partial A_z}{\partial r}\right|_{r=R^-} = \frac{4\pi}{c}\begin{cases} I_0 & 0<\phi<\pi \\ -I_0 & \pi<\phi<2\pi \end{cases}.$$

We therefore have

$$\sum_{m=0}^{\infty} mR^{m-1}\left(a_m\cos(m\phi)+b_m\sin(m\phi)\right) - \left(\frac{d_0}{R} - \sum_{m=0}^{\infty} m\frac{1}{R^{m+1}}\left(e_m\cos(m\phi)+f_m\sin(m\phi)\right)\right)$$
$$= \frac{4\pi}{c}\begin{cases} I_0 & 0<\phi<\pi \\ -I_0 & \pi<\phi<2\pi \end{cases},$$

and we can gather terms on the LHS and substitute e_m and f_m in terms of a_m and b_m, to obtain

$$-\frac{d_0}{R} + \sum_{m=0}^{\infty} 2mR^{m-1}\left[a_m\cos(m\phi)+b_m\sin(m\phi)\right] = \frac{4\pi}{c}\begin{cases} I_0 & 0<\phi<\pi \\ -I_0 & \pi<\phi<2\pi \end{cases}.$$

We use orthogonality to compute the coefficients,

$$2mR^{m-1}a_m\underbrace{\int_0^{2\pi}\cos(n\phi)\cos(m\phi)\,\mathrm{d}\phi}_{=\pi\delta_{mn}} = \frac{4\pi}{c}I_0\left(\underbrace{\int_0^{\pi}\cos(n\phi)}_{=0} - \underbrace{\int_\pi^{2\pi}\cos(n\phi)}_{=0}\right)\mathrm{d}\phi = 0,$$

$$2mR^{m-1}b_m\underbrace{\int_0^{2\pi}\sin(n\phi)\sin(m\phi)\,\mathrm{d}\phi}_{=\pi\delta_{mn}} = \frac{4\pi}{c}I_0\left(\underbrace{\int_0^{\pi}\sin(n\phi)}_{(1-(-1)^n)/n} - \underbrace{\int_\pi^{2\pi}\sin(n\phi)}_{(1-(-1)^n)/n}\right)\mathrm{d}\phi$$
$$\implies b_n = \frac{4I_0}{cn^2R^{n-1}}\left(1-(-1)^n\right)$$
$$|n=2k+1| \implies b_k = \frac{8I_0}{c\left(2k+1\right)^2R^{2k}}.$$

We therefore find that $a_m = e_m = 0$ for all m, while $f_k = R^{2(2k+1)}b_k \neq 0$ for all k. To find d_0, we integrate over ϕ and obtain

$$\int_0^{2\pi} \frac{d_0}{R}\,\mathrm{d}\phi = \frac{2\pi d_0}{R} = \sum_{m=0}^{\infty} 2mR^{m-1} b_m \underbrace{\int_0^{2\pi} \sin(m\phi)\,\mathrm{d}\phi}_{=0} - \frac{4\pi I_0}{c}\left(\int_0^{\pi} \mathrm{d}\phi - \int_{\pi}^{2\pi} \mathrm{d}\phi\right) = 0.$$

The potential is then

$$A_z(r,\phi) = \begin{cases} \sum_{k=0}^{\infty} r^{2k+1} \frac{4I_0}{c(2k+1)^2 R^{2k}} \sin((2k+1)\phi) & r < R \\ \sum_{k=0}^{\infty} \frac{1}{r^{2k+1}} \frac{4I_0 R^{2k+2}}{c(2k+1)^2} \sin((2k+1)\phi) & r > R \end{cases}$$

and so, we find the magnetic field

$$B_r(r,\phi) = \frac{1}{r}\frac{\partial A_z}{\partial \phi} = \begin{cases} \sum_{k=0}^{\infty} r^{2k} \frac{4I_0}{c(2k+1)R^{2k}} \cos((2k+1)\phi) & r < R \\ \sum_{k=0}^{\infty} \frac{1}{r^{2k+2}} \frac{4I_0 R^{2k+2}}{c(2k+1)} \cos((2k+1)\phi) & r > R \end{cases}$$

$$B_\phi(r,\phi) = -\frac{\partial A_z}{\partial r} = \begin{cases} \sum_{k=0}^{\infty} r^{2k} \frac{4I_0}{c(2k+1)R^{2k}} \sin((2k+1)\phi) & r < R \\ \sum_{k=0}^{\infty} \frac{1}{r^{2k+2}} \frac{4I_0 R^{2k+2}}{c(2k+1)} \sin((2k+1)\phi) & r > R \end{cases}$$

Dielectrics

In general, the charge density of a dielectric contains a couple of contributions,

$$\rho = \rho_{\text{ext}} + \rho_{\text{bound}},$$

where ρ_{bound} is the charge density of dipoles and ρ_{ext} are the rest of the free charges. For linear dielectrics, at the interface between two different materials with dielectric constants $\varepsilon^{\pm}$, the field components satisfy

$$\varepsilon^{+} E_{\perp}^{+} - \varepsilon^{-} E_{\perp}^{-} \equiv D_{\perp}^{+} - D_{\perp}^{-} = 4\pi\sigma_{\text{ext}},$$
$$E_{\parallel}^{+} = E_{\parallel}^{-},$$

where D is the electric displacement field, defined as

$$\vec{D} = \vec{E} + 4\pi\vec{P} \equiv \varepsilon\vec{E}.$$

The polarization field $\vec{P}$ of the bound charges (dipoles) in the dielectric material satisfies the Gauss law equivalent

$$\nabla \cdot \vec{P} = -\sigma_{\text{bound}}.$$

Problem 3

Consider two conducting, concentric spherical shells of radii a, b ($a < b$). The outer shell is charged $-q$ while the inner one is charged $+q$. The region between the shells is filled with a dielectric material, such that half of it has ε_1 and the other ε_2 (see figure 2).

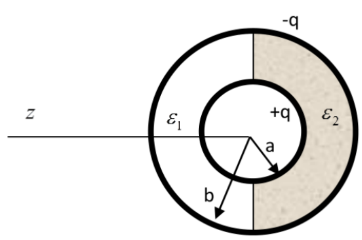

Fig. 2: Two concentric, conducting shells separated by a non-homogeneous dielectric.

1. Find the electric field.

2. Find the free and bound surface charge densities $\sigma_{\text{ext}}, \sigma_{\text{bound}}$ on the inner shell.

Solution

1. The shells are conductors, which means they are equipotential surfaces. We will obtain the electric field using Gauss for the electric displacement field $\vec{D}$,

$$\oiint \vec{D} \cdot \mathrm{d}\vec{S} = 4\pi Q_{\mathrm{tot}}.$$

From Gauss outside the outer sphere, the electric field outside vanishes. Inside the inner conducting sphere we also have a vanishing field, so we only need to solve for the field between the layers. Using the azimuthal symmetry we take the potential in the form

$$\Phi\left(a < r < b, \theta\right) = \sum_{\ell=0}^{\infty} \left(A_\ell r^\ell + B_\ell r^{-(\ell+1)}\right) P_\ell\left(\cos\theta\right).$$

We set the potential at $r \to \infty$ to zero and obtain that it must be zero on $r = b$, and therefore

$$B_\ell = A_\ell b^{2\ell+1}.$$

Inside the inner shell the potential is some constant Φ_0, and thus at $r = a$ we have

$$\Phi_0 = \sum_{\ell=0}^{\infty} A_\ell \left(a^\ell + b^{2\ell+1} a^{-(\ell+1)}\right) P_\ell\left(\cos\theta\right)$$
$$\implies A_0 = \frac{\Phi_0 a}{a+b}\ B_0 = \frac{\Phi_0 ab}{a+b},\ A_{\ell\neq 0} = B_{\ell\neq 0} = 0.$$

The potential is therefore

$$\Phi\left(a < r < b, \theta\right) = \frac{\Phi_0 a}{a+b}\left(1 + \frac{b}{r}\right).$$

The field is then

$$\vec{E} = -\frac{\partial\Phi}{\partial r}\hat{\mathbf{r}} - \frac{1}{r}\frac{\partial\Phi}{\partial\theta}\hat{\boldsymbol{\theta}} = \frac{\Phi_0 ab}{r^2\left(a+b\right)}\hat{\mathbf{r}} = \frac{B_0}{r^2}\hat{\mathbf{r}}.$$

We can find Φ_0 from Gauss law for the dielectric displacement, on some spherical surface with $a < r < b$,

$$\begin{aligned}
\oiint \vec{D} \cdot \mathrm{d}\vec{S} &= \oiint \vec{D}_1 \cdot \mathrm{d}\vec{S} + \oiint \vec{D}_2 \cdot \mathrm{d}\vec{S} \\
&= \int_0^{\pi/2} \mathrm{d}\theta \int_0^{2\pi} \mathrm{d}\varphi \varepsilon_1 \frac{B_0}{r^2} r^2 \sin\theta + \int_{\pi/2}^{\pi} \mathrm{d}\theta \int_0^{2\pi} \mathrm{d}\varphi \varepsilon_2 \frac{B_0}{r^2} r^2 \sin\theta \\
&= 2\pi B_0 \left(\varepsilon_1 + \varepsilon_2\right).
\end{aligned}$$

This surface contains in it a total charge q, and thus we find

$$2\pi B_0 (\varepsilon_1 + \varepsilon_2) = 4\pi q \implies \Phi_0 = \frac{2q(a+b)}{ab(\varepsilon_1+\varepsilon_2)},$$

and so

$$\vec{E} = \frac{2q}{(\varepsilon_1+\varepsilon_2)\, r^2}\hat{\mathbf{r}}.$$

2. The surface charge is attributed to the jump in the (perpendicular) displacement field,

$$4\pi\sigma_a = D_\perp(a_+) - \cancelto{0}{D_\perp(a_-)} = E_r \begin{cases} \varepsilon_1 & 0<\theta<\pi/2 \\ \varepsilon_2 & \pi/2<\theta<\pi \end{cases}$$

$$\implies \sigma_a = \frac{q}{2\pi a^2} \begin{cases} \frac{\varepsilon_1}{\varepsilon_1+\varepsilon_2} & 0<\theta<\pi/2 \\ \frac{\varepsilon_2}{\varepsilon_1+\varepsilon_2} & \pi/2<\theta<\pi \end{cases}.$$

Note that if $\varepsilon_1 = \varepsilon_2 = \varepsilon_0$, we would have the usual result for homogeneous surface charge on a sphere, $\sigma = q/4\pi a^2$. We can also check that our result yields the total charge q,

$$\int_0^{2\pi} \mathrm{d}\varphi \left[\int_0^{\pi/2} \sigma_a(a, 0<\theta<\pi/2) + \int_{\pi/2}^{\pi} \sigma_a(a, \pi/2<\theta<\pi)\right] a^2 \sin\theta\, \mathrm{d}\theta = q\left(\frac{\varepsilon_1+\varepsilon_2}{\varepsilon_1+\varepsilon_2}\right) = q.$$

The bound charge on the inner shell is found from the jump in $P_\perp$on $r=a$,

$$\sigma_{\text{bound}} = -\left(P_\perp(a_+) - P_\perp(a_-)\right),$$

and we use $4\pi\vec{P} = \vec{D} - \vec{E}$ to compute it. Inside the shell we have $P_\perp(a_-) = 0$, and outside we have

$$P_\perp(a_+) = \frac{q}{2\pi a^2(\varepsilon_1+\varepsilon_2)} \begin{cases} \varepsilon_1 - 1 & 0<\theta<\pi/2 \\ \varepsilon_2 - 1 & \pi/2<\theta<\pi \end{cases}$$

The bound charge is therefore

$$\sigma_{\text{bound}} = \frac{q}{2\pi a^2(\varepsilon_1+\varepsilon_2)} \begin{cases} 1-\varepsilon_1 & 0<\theta<\pi/2 \\ 1-\varepsilon_2 & \pi/2<\theta<\pi \end{cases}$$