Homework 5

- Throughout this homework set you are asked to solve the questions in Coulomb gauge.

Question 1

Find the magnetic field for a sphere with permeability constant μ in a homogeneous external magnetic field $B_0\hat{z}$.

Solution

There are no currents in the problem and thus $\vec{\nabla} \times \vec{B} = 0$ which means $\vec{B}$ is derived from some scalar potential $\vec{\nabla}\psi = \vec{B}$, such that

$$\nabla^2\psi = 0.$$

Far away from the sphere we have $\vec{B}\,(r \gg R) = B_0\hat{z}$ and $\psi\,(r \gg R) = B_0 z + C = B_0 r P_1\,(\cos\theta) + C$, and we can gauge $C = 0$. We solve the azimuthally symmetric Laplace equation for the magnetic potential using spherical separation of variables,

$$\psi = \begin{cases} \sum_{\ell=0}^{\infty} a_\ell r^\ell P_\ell\,(\cos\theta) & r < R \\ \sum_{\ell=0}^{\infty} \frac{b_\ell}{r^{\ell+1}} P_\ell\,(\cos\theta) + B_0 r P_1\,(\cos\theta) & r > R \end{cases}$$

where we used the $r \gg R$ limit to choose the form of the field outside the sphere. Using the continuity of $B_r = \partial\psi/\partial r$ across $r = R$, we find

$$\sum_{\ell=0}^{\infty} \ell a_\ell R^{\ell-1} P_\ell\,(\cos\theta) = -\sum_{\ell=0}^{\infty} (\ell+1)\,\frac{b_\ell}{R^{\ell+2}} P_\ell\,(\cos\theta) + B_0 P_1\,(\cos\theta)$$

$$\implies b_\ell = \begin{cases} \frac{R^3}{2}\,(B_0 - a_1) & \ell = 1 \\ -a_\ell R^{2\ell+1}\frac{\ell}{2(\ell+1)} & \ell \neq 1 \end{cases}$$

The tangent field

$$B_\theta = \frac{1}{r}\frac{\partial\psi}{\partial\theta}$$

must jump due to the difference in permeability, and thus ($\mu_0 = 1$)

$$\frac{1}{\mu}\frac{\partial\psi\left(R^-\right)}{\partial\theta} = \frac{\partial\psi(R^+)}{\partial\theta}.$$

Recall the definition of the associates Legendre polynomials, where

$$P_\ell^m\left(x\right) = \left(-1\right)^m\left(1-x^2\right)^{m/2}\frac{\mathrm{d}^m}{\mathrm{d}x^m}P_\ell\left(x\right).$$

In our case, the continuity conditions yields

$$\frac{1}{\mu}\sum_{\ell=0}^{\infty}a_\ell r^\ell\frac{\mathrm{d}}{\mathrm{d}x}P_\ell\left(x\right) = \frac{1}{\mu_0}\sum_{\ell=0}^{\infty}\frac{b_\ell}{r^{\ell+1}}\frac{\mathrm{d}}{\mathrm{d}x}P_\ell\left(x\right) + B_0 r\frac{\mathrm{d}}{\mathrm{d}x}P_1\left(x\right),$$

and thus only the $m=1$ associates Legendre polynomials are relevant here, and we can use their orthogonality. We have

$$\begin{aligned}\ell\neq1:\ b_\ell &= a_\ell\frac{\mu_0}{\mu}R^{2\ell+1},\\ \ell=1:\ b_1 &= R^3\left(\frac{1}{\mu}a_1 - B_0\right).\end{aligned}$$

In total, for $\ell=1$ we find

$$\begin{aligned}a_1 &= \mu B_0\left(\frac{3}{\mu+2}\right), b_1 = b_1 = R^3 B_0\left(\frac{1-\mu}{\mu+2}\right),\\ a_{\ell\neq1} &= b_{\ell\neq1} = 0,\end{aligned}$$

and in total,

$$\psi = \begin{cases}\frac{3\mu}{\mu+2}B_0 r\cos\theta & r<R\\ \left(\frac{1}{r^{\ell+1}}R^3 B_0\left(\frac{1-\mu}{\mu+2}\right) + B_0 r\right)\cos\theta & r>R\end{cases}$$

and

$$\vec{B} = \begin{cases}\frac{3\mu}{\mu+2}B_0\hat{z} & r<R\\ -\frac{1}{r^{\ell+2}}R^3 B_0\left(\frac{1-\mu}{\mu+2}\right)\left(\left(\ell+1\right)\cos\theta\hat{\mathbf{r}} + \sin\theta\hat{\boldsymbol{\theta}}\right) + B_0\hat{z} & r>R\end{cases}$$

Question 2

The surface charge density of a spherical shell of radius R is given by

$$\sigma_0(\theta) = \sigma_0 \cos\theta.$$

The sphere is rotating with constant angular velocity ω. Find the vector potential everywhere.

Hint: Use the result from the previous homework, where you found that

$$\frac{1}{|\vec{r} - \vec{r}'|} = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \frac{1}{2\ell+1} \frac{r_<^{\ell}}{r_>^{\ell+1}} Y_{\ell m}^{*}(\theta', \varphi') Y_{\ell m}(\theta, \varphi), \tag{1}$$

to compute $\vec{A}$ directly.

Solution

The surface density is

$$\vec{J} = v\sigma = \sigma_0 \omega R \sin\theta \cos\theta \hat{\boldsymbol{\varphi}}.$$

For later convenience, we transform the result to Cartesian coordinates, where

$$\hat{\boldsymbol{\varphi}} = -\hat{\mathbf{x}} \sin\varphi + \hat{\mathbf{y}} \cos\varphi.$$

We therefore have

$$\begin{aligned}
\vec{J} &= \sigma_0 \omega R \sin\theta \cos\theta \left[-\hat{\mathbf{x}} \sin\varphi + \hat{\mathbf{y}} \cos\varphi\right] \\
&= \sigma_0 \omega R \sin\theta \cos\theta \left[-\hat{\mathbf{x}} \frac{(e^{i\varphi} - e^{-i\varphi})}{2i} + \hat{\mathbf{y}} \frac{(e^{i\varphi} + e^{-i\varphi})}{2}\right] \\
&= \frac{1}{2} \sigma_0 \omega R \sin\theta \cos\theta \left[(\hat{\mathbf{y}} + i\hat{\mathbf{x}}) e^{i\varphi} + (\hat{\mathbf{y}} - i\hat{\mathbf{x}}) e^{-i\varphi}\right].
\end{aligned}$$

We solve the "Laplace" equation for the magnetic potential, where we note that the current is non vanishing only on the surface $r = R$,

$$\begin{aligned}
\vec{A}(\vec{r}) &= \frac{1}{c} \iiint \frac{\vec{J}(\vec{r}') \, \mathrm{d}^3\vec{r}'}{|\vec{r} - \vec{r}'|} \\
&= \frac{1}{c} \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \iint_{r=R} \mathrm{d}S' \vec{J}(\vec{r}') \frac{1}{2\ell+1} \frac{r_<^{\ell}}{r_>^{\ell+1}} Y_{\ell m}^{*}(\theta', \varphi') Y_{\ell m}(\theta, \varphi).
\end{aligned}$$

For $r < R$ we have

$$\vec{A}\left(r<R\right)=\frac{1}{c}\sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell}\iint \mathrm{d}S'\,\vec{J}\left(\vec{r}'\right)\frac{1}{2\ell+1}\frac{r^{\ell}}{R^{\ell+1}}Y_{\ell m}^{*}\left(\theta',\varphi'\right)Y_{\ell m}\left(\theta,\varphi\right)$$
$$=\frac{\sigma_0\omega}{2c}\sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell}\left(\frac{r}{R}\right)^{\ell}\frac{Y_{\ell m}\left(\theta,\varphi\right)}{2\ell+1}$$
$$\times\int_0^{2\pi}\int_0^{\pi}Y_{\ell m}^{*}\left(\theta',\varphi'\right)\sin^2\theta'\cos\theta'\left[\left(\hat{\mathbf{y}}+i\hat{\mathbf{x}}\right)e^{i\varphi}+\left(\hat{\mathbf{y}}-i\hat{\mathbf{x}}\right)e^{-i\varphi}\right]R^2\,\mathrm{d}\theta'\,\mathrm{d}\varphi'.$$

We use

$$\frac{1}{2}\sqrt{\frac{15}{2\pi}}e^{i\varphi}\sin\theta'\cos\theta'=Y_{2,1}\left(\theta',\varphi'\right),$$
$$-\frac{1}{2}\sqrt{\frac{15}{2\pi}}e^{-i\varphi}\sin\theta'\cos\theta'=Y_{2,-1}\left(\theta',\varphi'\right),$$

and thus the integrand becomes

$$2\sqrt{\frac{2\pi}{15}}Y_{\ell m}^{*}\left(\theta',\varphi'\right)\left[\left(\hat{\mathbf{y}}+i\hat{\mathbf{x}}\right)Y_{2,1}\left(\theta',\varphi'\right)-\left(\hat{\mathbf{y}}-i\hat{\mathbf{x}}\right)Y_{2,-1}\left(\theta',\varphi'\right)\right].$$

Using the orthogonality of the spherical harmonics, we obtain

$$\vec{A}\left(r<R\right)=\frac{\sigma_0\omega R^2}{2c}2\sqrt{\frac{2\pi}{15}}\sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell}\left(\frac{r}{R}\right)^{\ell}Y_{\ell m}\left(\theta,\varphi\right)\frac{\left(\hat{\mathbf{y}}+i\hat{\mathbf{x}}\right)\delta_{\ell,2}\delta_{m,1}-\left(\hat{\mathbf{y}}-i\hat{\mathbf{x}}\right)\delta_{\ell,2}\delta_{m,-1}}{2\ell+1}$$
$$=\frac{\sigma_0\omega r^2}{10c}2\sqrt{\frac{2\pi}{15}}\left[\left(\hat{\mathbf{y}}+i\hat{\mathbf{x}}\right)Y_{2,1}\left(\theta,\varphi\right)-\left(\hat{\mathbf{y}}-i\hat{\mathbf{x}}\right)Y_{2,-1}\left(\theta,\varphi\right)\right]$$
$$=\frac{\sigma_0\omega r^2}{5c}\sin\theta\cos\theta\left[\hat{\mathbf{y}}\frac{\left(e^{i\varphi}+e^{-i\varphi}\right)}{2}-\hat{\mathbf{x}}\frac{\left(e^{i\varphi}-e^{-i\varphi}\right)}{2i}\right]$$
$$=\frac{\sigma_0\omega r^2}{5c}\sin\theta\cos\theta\left[\hat{\mathbf{y}}\cos\varphi-\hat{\mathbf{x}}\sin\varphi\right]$$
$$=\frac{\sigma_0\omega r^2}{5c}\sin\theta\cos\theta\hat{\boldsymbol{\varphi}}.$$

A similar computation leads to

$$\vec{A}\left(r>R\right)=\frac{\sigma_0\omega R^5}{5cr^3}\sin\theta\cos\theta\hat{\boldsymbol{\varphi}}.$$

Question 3 - The classical electron model

Assume that the electron is a hollow conducting shell of radius R, charge uniformly with a total charge e, and rotating at some constant angular velocity ω.

1. Find the electric potential and field everywhere in space.

2. Show that the current density on the shell is

$$\vec{K}(\vec{r}) = \sigma\omega R\sin\theta\hat{\boldsymbol{\varphi}}.$$

3. Compute the vector potential. Show that the system has a magnetic dipole moment $\vec{m} = eR^2\vec{\omega}/(3c)$.
   Hint: Once again, use equation (1) to directly compute the potential.

4. Find the magnetic field $\vec{B}(\vec{r})$, and show that the magnetic field inside the shell is constant.

5. Compute the total energy stored in the EM field.

6. Compute the total angular momentum stored in the EM field.

7. Assume that the Electron's mass is related to its stored energy through $m = \mathcal{E}/c^2$ and that its angular momentum is $\hbar/2$. Under these assumption, evaluate R and the velocity $v = \omega R$. Do the values of R, v make sense?

Solution

1. This is a simple electrostatic problem, where we have

$$\Phi(\vec{r}) = \begin{cases} e/R & r < R \\ e/r & r > R \end{cases}$$

   and

$$\vec{E}(r) = \begin{cases} 0 & r < R \\ e\hat{\mathbf{r}}/r^2 & r > R \end{cases}$$

2. The surface charge density is given by

$$\vec{K}(r = R, \theta, \varphi) = v(\vec{r})\sigma,$$

where $\vec{v}$ is the velocity of points on the shell,

$$\begin{aligned}\vec{v} &= \vec{\omega} \times \vec{r}\big|_{r=R} \\ &= \omega \hat{\mathbf{z}} \times R\left(\sin\theta\cos\varphi\hat{\mathbf{x}} + \sin\theta\sin\varphi\hat{\mathbf{y}} + \cos\theta\hat{\mathbf{z}}\right) \\ &= \omega R\sin\theta\left(\cos\varphi\hat{\mathbf{y}} - \sin\varphi\hat{\mathbf{x}}\right) = \omega R\sin\theta\hat{\boldsymbol{\varphi}},\end{aligned}$$

and $\sigma = e/\left(4\pi R^2\right)$. We indeed find that

$$\vec{K}\left(r = R, \theta, \varphi\right) = \sigma\omega R\sin\theta\hat{\boldsymbol{\varphi}}.$$

3. We insert the Green's function definition into the integral and obtain

$$\begin{aligned}\vec{A}\left(\vec{r}\right) &= \frac{1}{c}\int_0^{2\pi}\int_0^{\pi} R^2\,\mathrm{d}\Omega' \frac{\vec{K}\left(\theta',\varphi'\right)}{\left|\vec{r}-\vec{r}'\right|} \\ &= \frac{\sigma\omega R^3}{c}\sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell}\frac{r_<^{\ell}}{r_>^{\ell+1}}\frac{Y_{\ell m}\left(\theta,\varphi\right)}{2\ell+1}\iint\left(\sin\theta'\cos\varphi'\hat{\mathbf{y}} - \sin\theta'\sin\varphi'\hat{\mathbf{x}}\right)Y_{\ell m}^*\left(\theta',\varphi'\right)\mathrm{d}\Omega'.\end{aligned}$$

We note that

$$\begin{aligned}-\sin\theta'\sin\varphi' &= -i\sqrt{\frac{2\pi}{3}}\left(Y_{1,1}+Y_{1,-1}\right) \\ \sin\theta'\cos\varphi' &= -\sqrt{\frac{2\pi}{3}}\left(Y_{1,1}-Y_{1,-1}\right),\end{aligned}$$

and use orthogonality to obtain

$$\begin{aligned}\vec{A}\left(\vec{r}\right) &= \frac{\sigma\omega R^3}{3c}\frac{r_<}{r_>^2}\left(-\sqrt{\frac{2\pi}{3}}\left(Y_{1,1}-Y_{1,-1}\right)\hat{\mathbf{y}} - i\sqrt{\frac{2\pi}{3}}\left(Y_{1,1}+Y_{1,-1}\right)\hat{\mathbf{x}}\right) \\ &= \frac{\sigma\omega R^3\sin\theta}{3c}\frac{r_<}{r_>^2}\left(-\sin\varphi\hat{\mathbf{x}} + \cos\varphi\hat{\mathbf{y}}\right) \\ &= \frac{\sigma\omega R^3\sin\theta}{3c}\frac{r_<}{r_>^2}\hat{\boldsymbol{\varphi}} \\ &= \frac{\sigma}{3c}\vec{\omega}\times\vec{r}\begin{cases} r/R & r<R \\ \left(R/r\right)^2 & r>R\end{cases}\end{aligned}$$

Using

$$\vec{m} = \frac{1}{2c}\iiint \vec{r}'\times\vec{J}\left(\vec{r}'\right)\mathrm{d}^3\vec{r}',$$

where

$$\vec{J} = \vec{K}\delta\left(r-R\right),$$

we obtain

$$\begin{aligned}
\vec{m} &= \frac{\sigma\omega R^4}{2c}\iint \sin\theta\left(\sin\theta\cos\varphi\hat{\mathbf{x}}+\sin\theta\sin\varphi\hat{\mathbf{y}}+\cos\theta\hat{\mathbf{z}}\right)\times\left(\cos\varphi\hat{\mathbf{y}}-\sin\varphi\hat{\mathbf{x}}\right)\mathrm{d}\Omega' \\
&= \frac{\sigma\omega R^4}{2c}\iint \sin\theta\left[\sin\theta\left(\cos^2\varphi+\sin^2\varphi\right)\hat{\mathbf{z}}-\left(\cos\theta\sin\varphi\hat{\mathbf{y}}+\cos\theta\cos\varphi\hat{\mathbf{x}}\right)\right]\mathrm{d}\Omega' \\
&= \frac{\sigma\omega R^4}{2c}\int_0^{\pi}\mathrm{d}\theta\left[2\pi\sin^3\theta\hat{\mathbf{z}}-\sin^2\theta\cos\theta\left(\hat{\mathbf{y}}\cancelto{0}{\int_0^{2\pi}\sin\varphi\,\mathrm{d}\varphi}+\hat{\mathbf{x}}\cancelto{0}{\int_0^{2\pi}\cos\varphi\,\mathrm{d}\varphi}\right)\right] \\
&= \frac{4\sigma\omega\pi R^4}{3c}\hat{\mathbf{z}}=\frac{eR^2\vec{\omega}}{3c}.
\end{aligned}$$

4. The magnetic field outside the shell is that of a dipole,

$$\begin{aligned}
\vec{B}\left(r>R\right) &= \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left(\frac{\sigma\omega R^3\sin^2\theta}{3c}\frac{R}{r^2}\right)\hat{\mathbf{r}}-\frac{1}{r}\frac{\partial}{\partial r}\left(\frac{\sigma\omega R^3\sin\theta}{3c}\frac{R}{r}\right)\hat{\boldsymbol{\theta}} \\
&= \frac{e\omega}{12\pi c}\frac{R^2}{r^3}\left[2\cos\theta\hat{\mathbf{r}}+\sin\theta\hat{\boldsymbol{\theta}}\right],
\end{aligned}$$

and inside we have

$$\begin{aligned}
\vec{B}\left(r<R\right) &= \frac{2\sigma\omega R}{3c}\left[\cos\theta\hat{\mathbf{r}}-\sin\theta\hat{\boldsymbol{\theta}}\right] \\
&= \frac{2\sigma\omega R}{3c}\left[\cos\theta\left(\sin\theta\cos\varphi\hat{\mathbf{x}}+\sin\theta\sin\varphi\hat{\mathbf{y}}+\cos\theta\hat{\mathbf{z}}\right)\right. \\
&\left.-\sin\theta\left(\cos\theta\cos\varphi\hat{\mathbf{x}}+\cos\theta\sin\varphi\hat{\mathbf{y}}-\sin\theta\hat{\mathbf{z}}\right)\right] \\
&= \frac{2\sigma\omega R}{3c}\hat{\mathbf{z}}=\frac{2e\omega}{12\pi cR}\hat{\mathbf{z}}
\end{aligned}$$

5. The energy density is given by

$$u_{em}=\frac{1}{8\pi}\left(E^2+B^2\right)=\frac{1}{8\pi}\begin{cases}\left(\frac{2e\omega}{12\pi cR}\right)^2 & r<R\\ \frac{e^2}{r^4}+\left(\frac{e\omega}{12\pi c}\frac{R^2}{r^3}\right)^2\left(1+3\cos^2\theta\right) & r>R\end{cases}$$

The total stored energy is then the volume integral of the density,

$$\begin{aligned}
U_{em} &= \int u_{em}\,\mathrm{d}V \\
&= \frac{1}{8\pi}\left[\left(\frac{2e\omega}{12\pi cR}\right)^2 4\pi\int_0^R r^2\,\mathrm{d}r+e^2 4\pi\int_R^\infty\frac{\mathrm{d}r}{r^2}+\left(\frac{e\omega R^2}{12\pi c}\right)^2\int_R^\infty\frac{\mathrm{d}r}{r^4}\left(4\pi+6\pi\int_0^\pi\cos^2\theta\sin\theta\,\mathrm{d}\theta\right)\right] \\
&= \frac{e^2}{8\pi}\left[\frac{3\omega^2 R}{54\pi c^2}+\frac{4\pi}{R}\right].
\end{aligned}$$

6. The momentum density in the field is given by

$$\vec{p}_{em} = \frac{1}{4\pi c}\left(\vec{E}\times\vec{B}\right) = \frac{1}{4\pi c}\begin{cases} 0 & r<R \\ \frac{e^2\omega}{12\pi c}\frac{R^2}{r^5}\sin\theta\hat{\boldsymbol{\varphi}} & r>R \end{cases}$$

and thus the angular momentum density outside the shell is

$$\vec{\ell} = \frac{\vec{r}}{4\pi c}\times\left(\vec{E}\times\vec{B}\right) = -\frac{e^2\omega}{48\pi^2c^2}\frac{R^2}{r^4}\sin\theta\hat{\boldsymbol{\theta}}.$$

Recall that

$$\hat{\boldsymbol{\theta}} = \cos\theta\cos\varphi\hat{\mathbf{x}} + \cos\theta\sin\varphi\hat{\mathbf{y}} - \sin\theta\hat{\mathbf{z}},$$

which means integration over φ will destroy the terms in the $\hat{\mathbf{x}},\hat{\mathbf{y}}$ directions and only $\hat{\mathbf{z}}$ will remain. The total stored angular momentum is then

$$\begin{aligned}\vec{L} &= \int \vec{\ell}\,\mathrm{d}V \\ &= 2\pi\int_0^\pi\int_R^\infty \frac{e^2\omega}{48\pi^2c^2}\frac{R^2}{r^2}\sin^3\theta\,\mathrm{d}r\,\mathrm{d}\theta\hat{\mathbf{z}} \\ &= \hat{\mathbf{z}}\frac{e^2\omega R}{18\pi c^2}.\end{aligned}$$

7. We equate the angular momentum we obtained to $\hbar/2$ and obtain the electron radius,

$$R_e = 4\pi\left(\frac{9c^2\hbar}{4e^2\omega}\right),$$

which means

$$v = \omega R_e = 4\pi\left(\frac{9c^2\hbar}{4e^2}\right) = 4\pi\left(\frac{9c}{4\alpha}\right),$$

where $\alpha = e^2/\hbar c$ is the fine structure constant. Since $\alpha \approx 1/137 \ll 1$, we obtain $v \gg c$! That is obviously impossible. Next, we compare the energies and get

$$m_ec^2 = \frac{e^2}{8\pi}\left[\frac{3\omega^2R}{54\pi c^2} + \frac{4\pi}{R}\right] = \frac{1}{8\pi}\frac{4e^4\omega}{9c^2\hbar}\left[1+\frac{9\hbar^2c^2}{8e^4}\right] = \frac{1}{4\pi}\frac{2e^2\alpha\omega}{9c}\left[1+\frac{9}{8\alpha^2}\right],$$

and in total we find

$$\omega = 4\pi m_ec^2\frac{9c}{2e^2\alpha\left[1+\frac{9}{8\alpha^2}\right]} \approx 16\pi\alpha m_e\frac{c^3}{e^2},$$

which gives a radius of

$$R_e = \frac{9\hbar}{16\alpha m_e c} \approx 10^{-5}cm,$$

which is obviously far larger than an atom, and therefore makes no sense. We find that the classical model fails to describe the real electron.