Class Exercise 6 - Dielectric materials, energy and momentum of the electromagnetic field

Dielectrics

In general, the charge density of a dielectric contains a couple of contributions,

$$\rho = \rho_{\text{ext}} + \rho_{\text{bound}},$$

where ρ_{bound} is the charge density of dipoles and ρ_{ext} are the rest of the free charges. For linear dielectrics, at the interface between two different materials with dielectric constants $\varepsilon^{\pm}$, the field components satisfy

$$\varepsilon^{+} E_{\perp}^{+} - \varepsilon^{-} E_{\perp}^{-} \equiv D_{\perp}^{+} - D_{\perp}^{-} = 4\pi\sigma_{\text{ext}},$$
$$E_{\parallel}^{+} = E_{\parallel}^{-},$$

where D is the electric displacement field, defined as

$$\vec{D} = \vec{E} + 4\pi\vec{P} \equiv \varepsilon\vec{E}.$$

The polarization field $\vec{P}$ of the bound charges (dipoles) in the dielectric material satisfies the Gauss law equivalent

$$\nabla \cdot \vec{P} = -\sigma_{\text{bound}}.$$

Problem 1

Consider two conducting, concentric spherical shells of radii a, b $(a < b)$. The outer shell is charged $-q$ while the inner one is charged $+q$. The region between the shells is filled with a dielectric material, such that half of it has ε_1 and the other ε_2 (see figure 2).

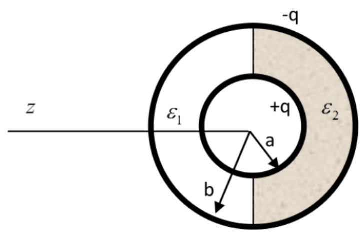

Fig. 1: Two concentric, conducting shells separated by a non-homogeneous dielectric.

1. Find the electric field.
2. Find the free and bound surface charge densities $\sigma_{\text{ext}}, \sigma_{\text{bound}}$ on the inner shell.

Solution

1. The shells are conductors, which means they are equipotential surfaces. We will obtain the electric field using Gauss for the electric displacement field $\vec{D}$,

$$\oint\!\!\!\oint \vec{D} \cdot \mathrm{d}\vec{S} = 4\pi Q_{\text{tot}}.$$

From Gauss outside the outer sphere, the electric field outside vanishes. Inside the inner conducting sphere we also have a vanishing field, so we only need to solve for the field between the layers. Using the azimuthal symmetry we take the potential in the form

$$\Phi\left(a < r < b, \theta\right) = \sum_{\ell=0}^{\infty} \left(A_\ell r^\ell + B_\ell r^{-(\ell+1)}\right) P_\ell\left(\cos\theta\right).$$

We set the potential at $r \to \infty$ to zero and obtain that it must be zero on $r = b$, and therefore

$$B_\ell = A_\ell b^{2\ell+1}.$$

Inside the inner shell the potential is some constant Φ_0, and thus at $r = a$ we have

$$\Phi_0 = \sum_{\ell=0}^{\infty} A_\ell \left(a^\ell + b^{2\ell+1} a^{-(\ell+1)}\right) P_\ell(\cos\theta)$$
$$\implies A_0 = \frac{\Phi_0 a}{a+b}\; B_0 = \frac{\Phi_0 ab}{a+b},\; A_{\ell\neq 0} = B_{\ell\neq 0} = 0.$$

The potential is therefore

$$\Phi(a < r < b, \theta) = \frac{\Phi_0 a}{a+b}\left(1 + \frac{b}{r}\right).$$

The field is then

$$\vec{E} = -\frac{\partial\Phi}{\partial r}\hat{\mathbf{r}} - \frac{1}{r}\frac{\partial\Phi}{\partial\theta}\hat{\boldsymbol{\theta}} = \frac{\Phi_0 ab}{r^2(a+b)}\hat{\mathbf{r}} = \frac{B_0}{r^2}\hat{\mathbf{r}}.$$

We can find Φ_0 from Gauss law for the dielectric displacement, on some spherical surface with $a < r < b$,

$$\begin{aligned}\oiint \vec{D}\cdot \mathrm{d}\vec{S} &= \oiint \vec{D}_1 \cdot \mathrm{d}\vec{S} + \oiint \vec{D}_2 \cdot \mathrm{d}\vec{S}\\ &= \int_0^{\pi/2}\mathrm{d}\theta \int_0^{2\pi}\mathrm{d}\varphi\varepsilon_1 \frac{B_0}{r^2} r^2\sin\theta + \int_{\pi/2}^{\pi}\mathrm{d}\theta\int_0^{2\pi}\mathrm{d}\varphi\varepsilon_2\frac{B_0}{r^2}r^2\sin\theta\\ &= 2\pi B_0(\varepsilon_1+\varepsilon_2).\end{aligned}$$

This surface contains in it a total charge q, and thus we find

$$2\pi B_0(\varepsilon_1+\varepsilon_2) = 4\pi q \implies \Phi_0 = \frac{2q(a+b)}{ab(\varepsilon_1+\varepsilon_2)},$$

and so

$$\vec{E} = \frac{2q}{(\varepsilon_1+\varepsilon_2)r^2}\hat{\mathbf{r}}.$$

2. The surface charge is attributed to the jump in the (perpendicular) displacement field,

$$4\pi\sigma_a = D_\perp(a_+) - \cancelto{0}{D_\perp(a_-)} = E_r \begin{cases}\varepsilon_1 & 0<\theta<\pi/2\\ \varepsilon_2 & \pi/2<\theta<\pi\end{cases}$$
$$\implies \sigma_a = \frac{q}{2\pi a^2}\begin{cases}\frac{\varepsilon_1}{\varepsilon_1+\varepsilon_2} & 0<\theta<\pi/2\\ \frac{\varepsilon_2}{\varepsilon_1+\varepsilon_2} & \pi/2<\theta<\pi\end{cases}.$$

Note that if $\varepsilon_1 = \varepsilon_2 = \varepsilon_0$, we would have the usual result for homogeneous surface charge on a sphere, $\sigma = q/4\pi a^2$. We can also check that our result yields the total charge q,

$$\int_0^{2\pi} \mathrm{d}\varphi \left[\int_0^{\pi/2} \sigma_a\left(a, 0<\theta<\pi/2\right) + \int_{\pi/2}^{\pi} \sigma_a\left(a, \pi/2<\theta<\pi\right)\right] a^2 \sin\theta \, \mathrm{d}\theta = q\left(\frac{\varepsilon_1+\varepsilon_2}{\varepsilon_1+\varepsilon_2}\right) = q.$$

The bound charge on the inner shell is found from the jump in $P_\perp$on $r = a$,

$$\sigma_{\text{bound}} = -\left(P_\perp\left(a_+\right) - P_\perp\left(a_-\right)\right),$$

and we use $4\pi\vec{P} = \vec{D} - \vec{E}$ to compute it. Inside the shell we have $P_\perp\left(a_-\right) = 0$, and outside we have

$$P_\perp\left(a_+\right) = \frac{q}{2\pi a^2\left(\varepsilon_1+\varepsilon_2\right)} \begin{cases} \varepsilon_1 - 1 & 0<\theta<\pi/2 \\ \varepsilon_2 - 1 & \pi/2<\theta<\pi \end{cases}$$

The bound charge is therefore

$$\sigma_{\text{bound}} = \frac{q}{2\pi a^2\left(\varepsilon_1+\varepsilon_2\right)} \begin{cases} 1-\varepsilon_1 & 0<\theta<\pi/2 \\ 1-\varepsilon_2 & \pi/2<\theta<\pi \end{cases}$$

Problem 2

An insulating ball of radius R has a constant dielectric constant ε. On the surface of the ball there is a charge distribution $\sigma = \sigma_0 \sin^2\theta$. Find the potential and electric field.

Solution

For any $r \neq R$, $\rho = 0$ and thus the potential solves the Laplace equation $\nabla^2\Phi = 0$. The setup is asimuthally symmetric and therefore $\Phi = \Phi(r,\theta)$ and

$$\Phi(r,\theta) = \begin{cases} \sum_{\ell=0}^{\infty} a_\ell r^\ell P_\ell(\cos\theta) & r < R \\ \sum_{\ell=0}^{\infty} b_\ell r^{-(\ell+1)} P_\ell(\cos\theta) & r > R \end{cases}$$

We use continuity across $r = R$ to determine that $b_\ell = a_\ell R^{2\ell+1}$. The charge distribution on the surface induces a jump of the electric field and therefore

$$\varepsilon_0 E_\perp\left(R^+\right) - \varepsilon E_\perp\left(R^-\right) = 4\pi\sigma_{\text{ext}}.$$

We take the derivative of the potential to obtain the field and find

$$\begin{aligned} -\varepsilon_0 \left.\frac{\partial\Phi}{\partial r}\right|_{r=R^+} + \varepsilon \left.\frac{\partial\Phi}{\partial r}\right|_{r=R^-} &= 4\pi\sigma_0\sin^2\theta \\ \implies \sum_{\ell=0}^{\infty}\left[\varepsilon\ell a_\ell R^{\ell-1} + (\ell+1)\,\varepsilon_0 a_\ell R^{2\ell+1}R^{-(\ell+2)}\right]P_\ell(\cos\theta) &= 4\pi\sigma_0\left(1-\cos^2\theta\right) \\ \implies \sum_{\ell=0}^{\infty} a_\ell R^{\ell-1}\left(\varepsilon\ell + \varepsilon_0(\ell+1)\right)P_\ell(\cos\theta) &= 4\pi\sigma_0\left(\frac{2}{3}P_0 - \frac{2}{3}P_2\right), \end{aligned}$$

where we recall that

$$P_0(x) = 1, P_2(x) = \frac{1}{2}\left(3x^2-1\right).$$

Using the orthogonality of the Legendre polynomials, we find that the only non-vanishing coefficients are

$$a_0 = \frac{8\pi\sigma_0 R}{3\varepsilon_0},\ b_0 = \frac{8\pi\sigma_0 R^2}{3\varepsilon_0},\ a_2 = -\frac{8\pi\sigma_0}{3R\left(2\varepsilon+3\varepsilon_0\right)},\ b_2 = -\frac{8\pi\sigma_0 R^4}{3\left(2\varepsilon+3\varepsilon_0\right)}.$$

The potential is then

$$\Phi(r,\theta) = \begin{cases} \frac{8\pi\sigma_0 R}{3\varepsilon_0} - \frac{4\pi\sigma_0}{3R(2\varepsilon+3\varepsilon_0)} r^2\left(3\cos^2\theta - 1\right) & r < R \\ \frac{8\pi\sigma_0 R^2}{3\varepsilon_0}\frac{1}{r} - \frac{4\pi\sigma_0 R^4}{3(2\varepsilon+3\varepsilon_0)}\frac{1}{r^3}\left(3\cos^2\theta - 1\right) & r > R \end{cases}$$

and the electric field is

$$E_r = -\frac{\partial \Phi}{\partial r} = \begin{cases} \frac{8\pi\sigma_0}{3R(2\varepsilon+3\varepsilon_0)} r \left(3\cos^2\theta - 1\right) & r < R \\ \frac{8\pi\sigma_0 R^2}{3\varepsilon_0}\frac{1}{r^2} - \frac{12\pi\sigma_0 R^4}{3(2\varepsilon+3\varepsilon_0)}\frac{1}{r^4}\left(3\cos^2\theta - 1\right) & r > R \end{cases}$$

$$E_\theta = -\frac{1}{r}\frac{\partial \Phi}{\partial \theta} = \begin{cases} \frac{4\pi\sigma_0}{R(2\varepsilon+3\varepsilon_0)} r \sin 2\theta & r < R \\ \frac{4\pi\sigma_0 R^4}{2\varepsilon+3\varepsilon_0}\frac{1}{r^4}\sin 2\theta & r > R \end{cases}$$

It is clear that $E_\theta\left(R^-\right) = E_\theta\left(R^+\right)$.

Problem 3

A very long solenoid of radius R, with n turns per unit length, carries a charge I. Outside and inside of the solenoid there are coaxial cylindrical shells with radii $a < b$ and length $\ell \gg b$. The inner cylinder is uniformly charged Q and the outer cylinder is uniformly charged $-Q$.

When the current on the solenoid is gradually turned off, the cylinders start rotating. Find the angular momentum before and after the current has been turned off.

Solution

Recall from class the Poynting vector

$$\vec{S} = \frac{c}{4\pi}\left(\vec{E}\times\vec{B}\right),$$

such that momentum of the fields is given by

$$\vec{P}_{em} = \frac{1}{c^2}\iiint \vec{S}\,\mathrm{d}V.$$

Therefore, the angular momentum will be

$$\vec{L} = \iiint \left(\vec{r}\times\vec{p}_{em}\right)\mathrm{d}V.$$

We therefore need to find the electric and magnetic fields. Since there is no explicit dependence on time, we have an electrostatic and magnetostatic problem.

Note that neither $\vec{E}$ nor $\vec{B}$ depend on z, θ because of the symmetry in the problem. $\vec{E}$ vanishes inside the inner conductor and outside the outer conductor, and outside we know that the magnetic field is effectively zero for a very long solenoid. The only contribution to the momentum comes from the region $a < r < R$.

We therefore obtain the electric field from Gauss,

$$\begin{aligned} a < r < b: \quad & \iint \vec{E}\cdot\mathrm{d}\vec{A} = 4\pi Q_{\text{tot}} \implies E_r\cdot 2\pi r\ell = 4\pi Q \\ & \implies \vec{E}\left(r\right) = \frac{2Q}{r\ell}\hat{\mathbf{r}}, \end{aligned}$$

and the magnetic field from Amper's law over a rectangular loop with vertical sides Δz on the solenoid,

$$\begin{aligned} a < r < R: \quad & \int \vec{B}\cdot\mathrm{d}\vec{\ell} = \frac{4\pi}{c}I_{\text{tot}} \implies B_z\Delta z = \frac{4\pi}{c}In\Delta z \\ & \implies \vec{B} = \frac{4\pi}{c}In\hat{\mathbf{z}}. \end{aligned}$$

The angular momentum at the beginning is then

$$\begin{aligned}\vec{L}_{em}(a<r<R) &= \frac{2Q}{c^2\ell} In \iiint \hat{\mathbf{r}} \times (\hat{\mathbf{r}} \times \hat{\mathbf{z}})\, r\, \mathrm{d}r\, \mathrm{d}\theta\, \mathrm{d}z \\ &= -\hat{\mathbf{z}} \frac{2Q}{c^2\ell} In \int_0^{\ell} \int_0^{2\pi} \int_a^R r\, \mathrm{d}r\, \mathrm{d}\theta\, \mathrm{d}z \\ &= -\hat{\mathbf{z}} \frac{2\pi Q}{c^2} In \left(R^2 - a^2\right).\end{aligned}$$

Now we compute the angular momentum after the current has been turned off. From Faraday's law we find

$$\oint \vec{E} \cdot \mathrm{d}\vec{\ell} = -\frac{1}{c}\frac{\partial}{\partial t} \int \vec{B} \cdot \mathrm{d}\vec{S},$$

which for a circular loop of radius $r > R$ gives

$$\begin{aligned}\int_0^{2\pi} E_\theta r\, \mathrm{d}\theta &= -\frac{1}{c}\frac{\partial}{\partial t} \int_0^{2\pi} \int_0^{r} B_z \Theta(R-r)\, r\, \mathrm{d}r\, \mathrm{d}\theta \\ \implies E_\theta &= -\frac{2\pi n R^2}{c^2 r} \frac{\partial I}{\partial t}.\end{aligned}$$

This field exerts a force on the outer cylinder,

$$\vec{F} = -Q\vec{E}(r=b) = \frac{2\pi Q n R^2}{c^2 b} \frac{\partial I}{\partial t} \hat{\boldsymbol{\theta}}.$$

The torque on the outer cylinder causes it to rotate,

$$\vec{\tau} = \vec{r} \times \vec{F} = b \frac{2\pi Q n R^2}{c^2 b} \frac{\partial I}{\partial t} \hat{\mathbf{r}} \times \hat{\boldsymbol{\theta}} = \frac{2\pi Q n R^2}{c^2} \frac{\partial I}{\partial t} \hat{\mathbf{z}}.$$

The angular momentum of the outer cylinder is then

$$\vec{L}_b = \int_{t_1}^{t_2} \vec{\tau}\, \mathrm{d}t = -\frac{2\pi Q n R^2}{c^2} I \hat{\mathbf{z}},$$

where at time t_1 there was current I and at time t_2 the current was completely turned off. A similar computation leads to

$$\vec{L}_a = +\frac{2\pi Q n a^2}{c^2} I \hat{\mathbf{z}}.$$

Is the angular momentum conserved? Of course!

$$\vec{L}_a + \vec{L}_b = -\hat{\mathbf{z}} \frac{2\pi Q}{c^2} In \left(R^2 - a^2\right) = \vec{L}_{em}.$$

Problem 4

Consider a ball with radius R and uniform charge Q. Find the force acting on the top half shell.

Solution

Recall from class that

$$\vec{F} = -\frac{1}{c^2}\frac{\partial}{\partial t}\iiint \vec{S}\,\mathrm{d}V + \oint \hat{T}\cdot \mathrm{d}\vec{A},$$

where $\hat{T}$ is the stress energy tensor, defined in Cartesian coordinates by

$$T_{ij} = \frac{1}{4\pi}\left(E_iE_j + B_iB_j - \frac{E^2+B^2}{2}\delta_{ij}\right).$$

Since $\vec{B}=0$ everywhere and the fields are static, we have $\vec{S}=0$ and $\partial\vec{S}/\partial t = 0$. The electric field is (Gauss)

$$\vec{E}\left(r\right) = \begin{cases} \frac{Qr}{R^3}\hat{\mathbf{r}} & r < R \\ \frac{Q}{r^2}\hat{\mathbf{r}} & r \geq R \end{cases}$$

and since $\hat{\mathbf{r}} = \sin\theta\cos\varphi\hat{\mathbf{x}} + \sin\theta\sin\varphi\hat{\mathbf{y}} + \cos\theta\hat{\mathbf{z}}$ we have

$$T = \frac{1}{4\pi}\begin{pmatrix} E_x^2 - \frac{E^2}{2} & E_xE_y & E_xE_z \\ E_yE_x & E_y^2 - \frac{E^2}{2} & E_yE_z \\ E_zE_x & E_zE_y & E_z^2 - \frac{E^2}{2} \end{pmatrix}.$$

Due to symmetry in φ, the force can only be in the $\hat{\mathbf{z}}$ direction - every contribution from the bottom hemisphere at some $\varphi = \varphi_0$ has an opposite and equal contribution from the point $\varphi = -\varphi_0$, which cancel the contribution of the radial (the cylindrical radial coordinate) and angular directions. However, we have broken symmetry in the $\hat{\mathbf{z}}$ axis by only considering one of the hemispheres, so the contributions in $\hat{\mathbf{z}}$ do not cancel.

In total, we obtain an integral for the force of the form

$$F_z = \oint \left(T_{xz}\,\mathrm{d}A_x + T_{yz}\,\mathrm{d}A_y + T_{zz}\,\mathrm{d}A_z\right),$$

where integration is done over the equator disk

$$A_{\text{disk}} = \left\{r,\theta,\varphi : 0 \leq r \leq R, \theta = \frac{\pi}{2}, 0 \leq \varphi \leq 2\pi\right\},$$
$$\mathrm{d}\vec{A}_{\text{disk}} = -r\,\mathrm{d}r\,\mathrm{d}\varphi\hat{\mathbf{z}},$$

and the top "bowl"

$$
\begin{aligned}
A_{\text{bowl}} &= \left\{ r, \theta, \varphi : r = R, 0 \le \theta \le \frac{\pi}{2}, 0 \le \varphi \le 2\pi \right\}, \\
\mathrm{d}\vec{A}_{\text{bowl}} &= R^2 \sin\theta \, \mathrm{d}\theta \, \mathrm{d}\varphi \hat{\mathbf{r}}.
\end{aligned}
$$

We integrate and find

$$
\begin{aligned}
F_z^{(\text{bowl})} &= \frac{1}{4\pi} \int_0^{2\pi} \int_0^{\pi/2} R^2 \sin\theta \, \mathrm{d}\theta \, \mathrm{d}\varphi \left[E_x E_z \sin\theta \cos\varphi + E_y E_z \sin\theta \sin\varphi + \left(E_z^2 - \frac{E^2}{2} \right) \cos\theta \right] \\
&= \frac{R^2}{4\pi} \left(\frac{Q}{R^2} \right)^2 \int_0^{2\pi} \int_0^{\pi/2} \sin\theta \cos\theta \, \mathrm{d}\theta \, \mathrm{d}\varphi \left[(\sin\theta \cos\varphi)^2 + (\sin\theta \sin\varphi)^2 + \left(\cos^2\theta - \frac{1}{2} \right) \right] \\
&= \frac{Q^2}{2R^2} \int_0^{\pi/2} \sin\theta \cos\theta \, \mathrm{d}\theta \left(\sin^2\theta + \cos^2\theta - \frac{1}{2} \right) \\
&= \frac{Q^2}{4R^2} \int_0^{\pi/2} \sin\theta \cos\theta \, \mathrm{d}\theta \\
&= \frac{Q^2}{8R^2} \int_0^{\pi/2} \sin 2\theta \, \mathrm{d}\theta \\
&= \frac{Q^2}{8R^2},
\end{aligned}
$$

$$
\begin{aligned}
F_z^{(\text{disk})} &= -\frac{1}{4\pi} \iint \left(E_z^2 - \frac{E^2}{2} \right) \mathrm{d}A_z \\
&= -\frac{1}{4\pi} \int_0^{2\pi} \int_0^R r \, \mathrm{d}r \, \mathrm{d}\varphi \left(E_z^2 - \frac{E^2}{2} \right) \\
&= -\frac{Q^2}{4\pi} \int_0^{2\pi} \int_0^R r \, \mathrm{d}r \, \mathrm{d}\varphi \frac{r^2}{R^6} \left(\cos^2\left(\frac{\pi}{2} \right) - \frac{1}{2} \right) \\
&= \frac{Q^2}{16R^2},
\end{aligned}
$$

which means

$$
\vec{F}_{\text{tot}} = \frac{3Q^2}{16R^2} \hat{\mathbf{z}}.
$$