Class Exercise 7 - The quasi-static approximation

Problem 1

Consider a ball with radius R and uniform charge Q. Find the force acting on the top half shell.

Solution

Recall from class that

$$\vec{F} = -\frac{1}{c^2}\frac{\partial}{\partial t}\iiint \vec{S}\,\mathrm{d}V + \oiint \hat{T}\cdot \mathrm{d}\vec{A},$$

where $\hat{T}$ is the stress tensor, defined in Cartesian coordinates by

$$T_{ij} = \frac{1}{4\pi}\left(E_iE_j + B_iB_j - \frac{E^2+B^2}{2}\delta_{ij}\right).$$

Since $\vec{B}=0$ everywhere and the fields are static, we have $\vec{S}=0$ and $\partial\vec{S}/\partial t = 0$. The electric field is (Gauss)

$$\vec{E}\left(r\right) = \begin{cases} \frac{Qr}{R^3}\hat{\mathbf{r}} & r < R \\ \frac{Q}{r^2}\hat{\mathbf{r}} & r \geq R \end{cases}$$

and since $\hat{\mathbf{r}} = \sin\theta\cos\varphi\hat{\mathbf{x}} + \sin\theta\sin\varphi\hat{\mathbf{y}} + \cos\theta\hat{\mathbf{z}}$ we have

$$T = \frac{1}{4\pi}\begin{pmatrix} E_x^2 - \frac{E^2}{2} & E_xE_y & E_xE_z \\ E_yE_x & E_y^2 - \frac{E^2}{2} & E_yE_z \\ E_zE_x & E_zE_y & E_z^2 - \frac{E^2}{2} \end{pmatrix}.$$

Due to symmetry in φ, the force can only be in the $\hat{\mathbf{z}}$ direction - every contribution from the bottom hemisphere at some $\varphi = \varphi_0$ has an opposite and equal contribution from the point $\varphi = -\varphi_0$, which cancel the contribution of the radial (the cylindrical radial coordinate) and angular directions. However, we have broken symmetry in the $\hat{\mathbf{z}}$ axis by only considering one of the hemispheres, so the contributions in $\hat{\mathbf{z}}$ do not cancel.

In total, we obtain an integral for the force in the form

$$F_z = \oint \left(T_{xz}\,\mathrm{d}A_x + T_{yz}\,\mathrm{d}A_y + T_{zz}\,\mathrm{d}A_z\right),$$

where integration is done over the equator disk

$$\begin{aligned} A_{\text{disk}} &= \left\{r,\theta,\varphi : 0 \le r \le R, \theta = \frac{\pi}{2}, 0 \le \varphi \le 2\pi\right\}, \\ \mathrm{d}\vec{A}_{\text{disk}} &= -r\,\mathrm{d}r\,\mathrm{d}\varphi\hat{\mathbf{z}}, \end{aligned}$$

and the top "bowl"

$$\begin{aligned} A_{\text{bowl}} &= \left\{r,\theta,\varphi : r = R, 0 \le \theta \le \frac{\pi}{2}, 0 \le \varphi \le 2\pi\right\}, \\ \mathrm{d}\vec{A}_{\text{bowl}} &= R^2 \sin\theta\,\mathrm{d}\theta\,\mathrm{d}\varphi\hat{\mathbf{r}}. \end{aligned}$$

We integrate and find

$$\begin{aligned} F_z^{(\text{bowl})} &= \frac{1}{4\pi}\int_0^{2\pi}\int_0^{\pi/2} R^2\sin\theta\,\mathrm{d}\theta\,\mathrm{d}\varphi\left[E_xE_z\sin\theta\cos\varphi + E_yE_z\sin\theta\sin\varphi + \left(E_z^2 - \frac{E^2}{2}\right)\cos\theta\right] \\ &= \frac{R^2}{4\pi}\left(\frac{Q}{R^2}\right)^2\int_0^{2\pi}\int_0^{\pi/2}\sin\theta\cos\theta\,\mathrm{d}\theta\,\mathrm{d}\varphi\left[(\sin\theta\cos\varphi)^2 + (\sin\theta\sin\varphi)^2 + \left(\cos^2\theta - \frac{1}{2}\right)\right] \\ &= \frac{Q^2}{2R^2}\int_0^{\pi/2}\sin\theta\cos\theta\,\mathrm{d}\theta\left(\sin^2\theta + \cos^2\theta - \frac{1}{2}\right) \\ &= \frac{Q^2}{4R^2}\int_0^{\pi/2}\sin\theta\cos\theta\,\mathrm{d}\theta = \frac{Q^2}{8R^2}\int_0^{\pi/2}\sin 2\theta\,\mathrm{d}\theta = \frac{Q^2}{8R^2}, \end{aligned}$$

$$\begin{aligned} F_z^{(\text{disk})} &= -\frac{1}{4\pi}\iint\left(E_z^2 - \frac{E^2}{2}\right)\mathrm{d}A_z \\ &= -\frac{1}{4\pi}\int_0^{2\pi}\int_0^R r\,\mathrm{d}r\,\mathrm{d}\varphi\left(E_z^2 - \frac{E^2}{2}\right) \\ &= -\frac{Q^2}{4\pi}\int_0^{2\pi}\int_0^R r\,\mathrm{d}r\,\mathrm{d}\varphi\frac{r^2}{R^6}\left(\cos^2\left(\frac{\pi}{2}\right) - \frac{1}{2}\right) \\ &= \frac{Q^2}{16R^2}, \end{aligned}$$

which gives a total force

$$\vec{F}_{\text{tot}} = \frac{3Q^2}{16R^2}\hat{\mathbf{z}}.$$

Problem 2

An electric field with frequency ω propagates along the axis of a conducting cylinder with conductivity σ. Describe the electric field in the conductor in the quasi-static approximation.

Solution

Before using the quasi-static approximation, we combine the pair of Maxwell equations

$$\vec{\nabla} \times \vec{B} = \frac{4\pi}{c}\vec{J} + \frac{1}{c}\frac{\partial \vec{E}}{\partial t},$$

and

$$\vec{\nabla} \times \vec{E} = -\frac{1}{c}\frac{\partial \vec{B}}{\partial t},$$

into a single differential equation for $\vec{E}$. We take the rotor of the second equation and insert into it the first one,

$$\vec{\nabla} \times \left(\vec{\nabla} \times \vec{E}\right) = -\frac{1}{c}\frac{\partial}{\partial t}\left(\vec{\nabla} \times \vec{B}\right) = -\frac{4\pi\sigma}{c^2}\frac{\partial}{\partial t}\vec{E} - \frac{1}{c^2}\frac{\partial^2 \vec{E}}{\partial t^2},$$

where we used the relation $\vec{J} = \sigma\vec{E}$. Neglecting the charge density (due to super fast decay in the large σ limit), we can use

$$\vec{\nabla} \times \left(\vec{\nabla} \times \vec{E}\right) = \vec{\nabla}\left(\underbrace{\vec{\nabla} \cdot \vec{E}}_{=0}\right) - \nabla^2\vec{E},$$

and therefore we have

$$\nabla^2\vec{E} = \frac{4\pi\sigma}{c^2}\frac{\partial}{\partial t}\vec{E} + \frac{1}{c^2}\frac{\partial^2 \vec{E}}{\partial t^2}.$$

Assuming $\vec{E} \propto e^{-i\omega t}\hat{\boldsymbol{\epsilon}}$, we find

$$\nabla^2\vec{E} = \left(-\frac{4\pi i\omega\sigma}{c^2} - \frac{\omega^2}{c^2}\right)\vec{E}.$$

Now we apply the quasi-static approximation $\sigma \gg \omega$, and find that the second term is negligible. In cylindrical coordinates we find

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \vec{E}}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 \vec{E}}{\partial \theta^2} + \frac{\partial^2 \vec{E}}{\partial z^2} = -\frac{4\pi i\omega\sigma}{c^2}\vec{E}.$$

As usual, we look for solutions of the form

$$E(\vec{r}) = R(r)\, e^{im\theta} e^{ikz},$$

where the full solution will be $E(\vec{r})\, e^{-i\omega t}$. Inserting this form into the above equation gives a Bessel equation,

$$\frac{\partial^2 R(r)}{\partial^2 r} + \frac{1}{r}\frac{\partial R(r)}{\partial r} + \left(\left(\sqrt{\frac{4\pi i\omega\sigma}{c^2} - k^2}\right)^2 - \frac{m^2}{r^2}\right) R(r) = 0.$$

The only solution which does not diverge at $r = 0$ is a Bessel function of the first kind, and therefore inside the cylinder we have

$$E(\vec{r}, t) = e^{-i\omega t} \sum_{m,k} A_{m,k} J_m \left(r\sqrt{\frac{4\pi i\omega\sigma}{c^2} - k^2}\right) e^{im\theta} e^{ikz}.$$

For simplicity, we consider now a cylinder with complete rotational and translational symmetries, such that the problem is independent of both z and θ. Therefore, we focus on the $m = k = 0$ modes, for which the solution becomes

$$E(\vec{r}, t) \propto J_0\left(\sqrt{\frac{4\pi i\omega\sigma}{c^2}} r\right) e^{-i\omega t}.$$

For $(r\sqrt{\sigma\omega}/c) \gg 1$, the dominant contribution from the Bessel function J_0 is

$$J_0(s \gg 1) \sim e^{is} \implies J_0\left(\sqrt{\frac{4\pi i\omega\sigma}{c^2}} r\right) \sim \exp\left(i\sqrt{\frac{4\pi i\omega\sigma}{c^2}} r\right).$$

Recall that $\sqrt{i} = (1+i)/\sqrt{2}$, and thus

$$J_0\left(\sqrt{\frac{4\pi i\omega\sigma}{c^2}} r\right) \sim \exp\left(i\frac{(1+i)}{c}\sqrt{2\pi\omega\sigma} r\right) = \underbrace{\exp\left(i\frac{\sqrt{2\pi\omega\sigma}}{c} r\right)}_{\text{oscillatory}} \underbrace{\exp\left(-\frac{\sqrt{2\pi\omega\sigma}}{c} r\right)}_{\text{decaying}},$$

which means the field decays inside the conductor! Notice that as ω or σ grow, the decay gets prominent at increasingly short distances. At the limit of infinite conductivity the field inside the conductor vanishes completely.

Problem 3

An infinitely long metal cylinder with radius a, conductivity σ and magnetic permeability μ is coaxial with an infinite solenoid of radius $b > a$ and n turns per unit length, which carries a current $I = I_0 e^{-i\omega t}$. Find the magnetic and electric fields everywhere in the quasi-static approximation.

Solution

The system is symmetric with respect to the axis of the cylinder (which we choose to be $\hat{\mathbf{z}}$), and the only variation of the current is in time. The primary magnetic field H_0, due to the solenoid, is therefore uniform in space and directed along the axis of symmetry. In turn, H_0 will induce an electric field on the cylinder (Faraday's),

$$\oint \vec{E} \cdot \mathrm{d}\vec{\ell} = -\frac{1}{c}\frac{\partial}{\partial t}\int \vec{B} \cdot \mathrm{d}\vec{S},$$

producing, in turn, currents in the $\hat{\boldsymbol{\theta}}$ direction all along the cylinder, since

$$\vec{J} = \sigma \vec{E} = \sigma E_\theta \hat{\boldsymbol{\theta}}.$$

We find that the solenoid has induced a current configuration (termed *eddy currents*) which look like many additional coaxial solenoids inside the cylinder!

The "inner solenoids" contribute only to the magnetic field inside the cylinder, since outside a solenoid the magnetic field is zero. Thus, the total magnetic field outside the cylinder is $H_0 = B_0/\mu = nI_0 e^{-i\omega t}$, while inside it is given by the solution to

$$\nabla^2 H = \frac{4\pi\mu\sigma}{c^2}\frac{\partial H}{\partial t},$$

gotten by using the quasi-static approximation of $\partial E/\partial t \propto \omega E \ll \sigma E$ in the Maxwell equations

$$\vec{\nabla} \times \vec{B} = \frac{4\pi\mu}{c}\vec{J} + \frac{1}{c}\frac{\partial \vec{E}}{\partial t},$$

and

$$\vec{\nabla} \times \vec{E} = -\frac{1}{c}\frac{\partial \vec{B}}{\partial t}.$$

We know that the magnetic field will only be in the $\hat{\mathbf{z}}$ direction, and due to symmetry there will be no dependence on θ and z. Therefore, we obtain a Bessel equation for H in the form

$$\left[\frac{\mathrm{d}^2}{\mathrm{d}r^2} + \frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r} + k^2\right] H = 0,$$

where

$$k = \frac{1+i}{\delta},\ \delta = \frac{c}{(2\pi\mu\sigma\omega)^{1/2}},\ H = H_z(r)\, e^{-i\omega t},\ H_\theta = H_r = 0.$$

The boundary condition at $r = a$ is $H(a) = H_0$, so the only finite (for $r \to 0$) solution is

$$H = H_0 \frac{J_0(kr)}{J_0(ka)},$$

where $J_0(kr)$ is the zero-order Bessel function of the first kind. Outside the cylinder we have $H(a \le r \le b) = H_0$ and $H(r > b) = 0$. We find the electric field inside the cylinder using Faraday's law,

$$\oint \vec{E} \cdot \mathrm{d}\vec{\ell} = -\frac{1}{c}\frac{\partial}{\partial t}\int \vec{B} \cdot \mathrm{d}\vec{S} = -\frac{\mu}{c}\frac{\partial}{\partial t}\int \vec{H} \cdot \mathrm{d}\vec{S}$$
$$\Longrightarrow\ 2\pi r E_\theta = \frac{2\pi\mu i\omega}{c}\frac{H_0 e^{-i\omega t}}{J_0(ka)}\int_0^r J_0(kr')\, r'\, \mathrm{d}r'$$

We use the integral identity for Bessel functions

$$\int x^p J_{p-1}(x)\, \mathrm{d}x = x^p J_p(x) + c,$$

to obtain

$$\int_0^r J_0(kr')\, r'\, \mathrm{d}r' = \frac{1}{k^2}\left[kr J_1(kr) - (k \cdot 0)\, J_1(0)\right] = \frac{r}{k} J_1(kr),$$

where we have used the fact that $J_{n>0}(0) = 0$. The electric field inside the cylinder is then

$$2\pi r E_\theta = \frac{2\pi\mu i\omega H_0 e^{-i\omega t}}{ck}\frac{r J_1(kr)}{J_0(ka)}$$
$$\Longrightarrow E_\theta(r < a) = \frac{i\omega\mu H_0 e^{-i\omega t}}{ck}\frac{J_1(kr)}{J_0(ka)}.$$

The current density is then given by

$$J_\theta = \frac{i\omega\mu H_0 e^{-i\omega t}}{\sigma ck}\frac{J_1(kr)}{J_0(ka)},\quad J_r = J_z = 0.$$

To determine the electric field outside the cylinder we use Faraday's law once more, and find

$$\oint \vec{E}\cdot \mathrm{d}\vec{\ell} = -\frac{\mu}{c}\frac{\partial}{\partial t}\int \vec{H}_{\mathrm{tot}}\cdot \mathrm{d}\vec{S}$$

$$\implies E_\theta = \frac{2\pi}{2\pi r}\frac{i\mu\omega}{c}e^{-i\omega t}\int_0^r \left[H_0\frac{J_0\left(kr'\right)}{J_0\left(ka\right)}\Theta\left(a-r'\right) + H_0\left[\Theta\left(b-r'\right)-\Theta\left(a-r'\right)\right]\right] r'\,\mathrm{d}r'$$

$$\implies E_\theta\left(a\le r\le b\right) = \left[\frac{i\mu\omega a H_0}{crk}\frac{J_1\left(ka\right)}{J_0\left(ka\right)} + \frac{i\mu\omega H_0}{cr}\left(r^2-a^2\right)\right]e^{-i\omega t},$$

$$\implies E_\theta\left(r>b\right) = \left[\frac{i\mu a\omega H_0}{crk}\frac{J_1\left(ka\right)}{J_0\left(ka\right)} + \frac{i\mu\omega H_0}{cr}\left(b^2-a^2\right)\right]e^{-i\omega t}.$$