Homework 6

Question 1

Consider an Iron sphere of radius R that carries a charge Q and a uniform magnetization $\vec{M} = M\hat{\mathbf{z}}$. The sphere is initially at rest.

1. Find the electric and magnetic fields.
2. Compute the angular momentum stored in the electromagnetic fields.
3. Suppose the sphere is gradually (and uniformly) demagnetized (perhaps by heating it up). Find the total angular momentum imparted to the sphere in the course of the demagnetization.
4. Suppose instead of demagnetizing the sphere we discharge it, by connecting a grounding wire to the north pole. Assume the current flows over the surface in such a way that the charge density remains uniform. Calculate the angular momentum imparted to the sphere in the course of the discharge. **Hint**: Recall that the Lorentz force on a single charged particle is $\vec{F} = q\vec{E} + q\vec{v} \times \vec{B}$.

Solution

1. Since the sphere is a conductor, inside $\vec{E}(r < R) = 0$ and from Gauss we have $\vec{E}(r > R) = Q/(4\pi r^3)\,\hat{\mathbf{r}}$. The vector potential satisfies

$$\vec{A}(\vec{r}) = \frac{1}{c}\iiint \frac{\vec{J}_b(\vec{r}')}{|\vec{r} - \vec{r}'|}\,\mathrm{d}^3\vec{r}',$$

where here we have a surface current due to the magnetization (as seen in the lectures)

$$\begin{aligned}
\vec{J}(\vec{r}') &= c\vec{M} \times \hat{\mathbf{r}}\delta(r - R) \\
&= cM\sin\theta\delta(r - R)\,\hat{\boldsymbol{\varphi}} \\
&= \frac{1}{2}cM\sin\theta\delta(r - R)\left[(\hat{\mathbf{y}} + i\hat{\mathbf{x}})\,e^{i\varphi} + (\hat{\mathbf{y}} - i\hat{\mathbf{x}})\,e^{-i\varphi}\right]
\end{aligned}$$

The bound current density is zero since M is a constant, so

$$\vec{J}_b = c\vec{\nabla} \times \vec{M} = 0.$$

Therefore, using the spherical symmetry of the setup we have

$$\begin{aligned}
\vec{A}\left(\vec{r}\right) &= M \iint \frac{\sin\theta'}{\left|\vec{r}-\vec{r}'\right|}\hat{\boldsymbol{\varphi}}\,\mathrm{d}\Omega \\
&= \frac{1}{2} M Y_{\ell m}\left(\theta,\varphi\right) \sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell} \iint_{r=R} \mathrm{d}S' \sin\theta' \frac{\left[\left(\hat{\mathbf{y}}+i\hat{\mathbf{x}}\right)e^{i\varphi}+\left(\hat{\mathbf{y}}-i\hat{\mathbf{x}}\right)e^{-i\varphi}\right]}{2\ell+1}\frac{r_<^\ell}{r_>^{\ell+1}} Y^*_{\ell m}\left(\theta',\varphi'\right) \\
&= \frac{1}{2} M R^2 \sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell} \frac{r_<^\ell Y_{\ell m}\left(\theta,\varphi\right)}{r_>^{\ell+1}} \iint_{r=R} \mathrm{d}S' 2\sqrt{\frac{2\pi}{3}} \frac{\left[-\left(\hat{\mathbf{y}}+i\hat{\mathbf{x}}\right)Y_{1,1}+\left(\hat{\mathbf{y}}-i\hat{\mathbf{x}}\right)Y_{1,-1}\right]}{2\ell+1} Y^*_{\ell m}\left(\theta',\varphi'\right) \\
&= \frac{1}{2} M R^2 \sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell} \frac{r_<^\ell Y_{\ell m}\left(\theta,\varphi\right)}{r_>^{\ell+1}} 2\sqrt{\frac{2\pi}{3}} \frac{\left[-\left(\hat{\mathbf{y}}+i\hat{\mathbf{x}}\right)\delta_{\ell,1}\delta_{m,1}+\left(\hat{\mathbf{y}}-i\hat{\mathbf{x}}\right)\delta_{\ell,1}\delta_{m,-1}\right]}{2\ell+1} \\
&= \frac{1}{2}\frac{MR^2}{3} 2\sqrt{\frac{2\pi}{3}} \frac{r_<}{r_>^2}\left[-\left(\hat{\mathbf{y}}+i\hat{\mathbf{x}}\right)Y_{1,1}+\left(\hat{\mathbf{y}}-i\hat{\mathbf{x}}\right)Y_{1,-1}\right] \\
&= \frac{1}{2}\frac{MR^2}{3}\frac{r_<}{r_>^2}\sin\theta\left[\left(\hat{\mathbf{y}}+i\hat{\mathbf{x}}\right)\sin\theta e^{i\varphi}+\left(\hat{\mathbf{y}}-i\hat{\mathbf{x}}\right)\sin\theta e^{-i\varphi}\right] \\
&= \frac{MR^2}{3}\frac{r_<}{r_>^2}\sin\theta\hat{\boldsymbol{\varphi}},
\end{aligned}$$

and thus

$$\vec{A} = \begin{cases} \frac{Mr}{3}\sin\theta\hat{\boldsymbol{\varphi}} & r<R \\ \frac{MR^3}{3r^2}\sin\theta\hat{\boldsymbol{\varphi}} & r>R \end{cases}$$

and

$$\begin{aligned}
\vec{B} &= \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left(A_\varphi\sin\theta\right)\hat{\mathbf{r}} - \frac{1}{r}\frac{\partial}{\partial r}\left(rA_\varphi\right)\hat{\boldsymbol{\theta}} \\
&= \begin{cases} \frac{2M}{3}\left[\cos\theta\hat{\mathbf{r}}-\sin\theta\hat{\boldsymbol{\theta}}\right] = \frac{2M}{3}\hat{\mathbf{z}} & r<R \\ \frac{MR^3}{3r^3}\left[2\cos\theta\hat{\mathbf{r}}+\sin\theta\hat{\boldsymbol{\theta}}\right] & r>R \end{cases}
\end{aligned}$$

(a)

$$\mathbf{E} = \left\{ \begin{array}{ll} 0, & (r < R) \\ \dfrac{1}{4\pi\epsilon_0}\dfrac{Q}{r^2}\,\hat{\mathbf{r}}, & (r > R) \end{array} \right\}; \quad \mathbf{B} = \left\{ \begin{array}{ll} \frac{2}{3}\mu_0 M\,\hat{\mathbf{z}}, & (r < R) \\ \dfrac{\mu_0}{4\pi}\dfrac{m}{r^3}\left[2\cos\theta\,\hat{\mathbf{r}} + \sin\theta\,\hat{\boldsymbol{\theta}}\right], & (r > R) \end{array} \right\} \quad \text{(Ex. 6.1)}$$

(where $m = \frac{4}{3}\pi R^3 M$); $\wp = \epsilon_0(\mathbf{E}\times\mathbf{B}) = \frac{\mu_0}{(4\pi)^2}\frac{Qm}{r^5}(\hat{\mathbf{r}}\times\hat{\boldsymbol{\theta}})\sin\theta$, and $(\hat{\mathbf{r}}\times\hat{\boldsymbol{\theta}}) = \hat{\boldsymbol{\phi}}$, so

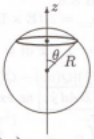
$$\boldsymbol{\ell} = \mathbf{r}\times\wp = \frac{\mu_0}{(4\pi)^2}\frac{mQ}{r^4}\sin\theta(\hat{\mathbf{r}}\times\hat{\boldsymbol{\phi}}).$$

But $(\hat{\mathbf{r}}\times\hat{\boldsymbol{\phi}}) = -\hat{\boldsymbol{\theta}}$, and only the z component will survive integration, so (since $(\hat{\boldsymbol{\theta}})_z = -\sin\theta$):

$$\mathbf{L} = \frac{\mu_0 mQ}{(4\pi)^2}\,\hat{\mathbf{z}}\int\frac{\sin^2\theta}{r^4}\left(r^2\sin\theta\,dr\,d\theta\,d\phi\right). \quad \int_0^{2\pi} d\phi = 2\pi; \quad \int_0^{\pi}\sin^3\theta\,d\theta = \frac{4}{3}; \quad \int_R^{\infty}\frac{1}{r^2}\,dr = \left(-\frac{1}{r}\right)\Big|_R^{\infty} = \frac{1}{R}.$$

$$\mathbf{L} = \frac{\mu_0 mQ}{(4\pi)^2}\,\hat{\mathbf{z}}\,(2\pi)\left(\frac{4}{3}\right)\left(\frac{1}{R}\right) = \boxed{\frac{2}{9}\mu_0 MQR^2\,\hat{\mathbf{z}}.}$$

(b) Apply Faraday's law to the ring shown:



$$\oint\mathbf{E}\cdot d\mathbf{l} = E(2\pi r\sin\theta) = -\frac{d\Phi}{dt} = -\pi(r\sin\theta)^2\left(\frac{2}{3}\mu_0\frac{dM}{dt}\right)$$

$$\Rightarrow \boxed{\mathbf{E} = -\frac{\mu_0}{3}\frac{dM}{dt}(r\sin\theta)\,\hat{\boldsymbol{\phi}}.}$$

The force on a patch of surface (da) is $d\mathbf{F} = \sigma\mathbf{E}\,da = -\frac{\mu_0\sigma}{3}\frac{dM}{dt}(r\sin\theta)\,da\,\hat{\boldsymbol{\phi}}\ \left(\sigma = \frac{Q}{4\pi R^2}\right)$.
The torque on the patch is $d\mathbf{N} = \mathbf{r}\times d\mathbf{F} = -\frac{\mu_0\sigma}{3}\frac{dM}{dt}\left(r^2\sin\theta\right)da\,(\hat{\mathbf{r}}\times\hat{\boldsymbol{\phi}})$. But $(\hat{\mathbf{r}}\times\hat{\boldsymbol{\phi}}) = -\hat{\boldsymbol{\theta}}$, and we want only the z component $(\hat{\boldsymbol{\theta}}_z = -\sin\theta)$:

$$\mathbf{N} = -\frac{\mu_0\sigma}{3}\frac{dM}{dt}\,\hat{\mathbf{z}}\int r^2\sin^2\theta\left(r^2\sin\theta\,d\theta\,d\phi\right).$$

Here $r = R$; $\int_0^{\pi}\sin^3\theta\,d\theta = \frac{4}{3}$; $\int_0^{2\pi}d\phi = 2\pi$, so $\mathbf{N} = -\frac{\mu_0\sigma}{3}\frac{dM}{dt}\,\hat{\mathbf{z}}\,R^4\left(\frac{4}{3}\right)(2\pi) = \boxed{-\frac{2\mu_0}{9}QR^2\frac{dM}{dt}\,\hat{\mathbf{z}}.}$

$$\mathbf{L} = \int\mathbf{N}\,dt = -\frac{2\mu_0}{9}QR^2\,\hat{\mathbf{z}}\int_M^0 dM = \boxed{\frac{2\mu_0}{9}MQR^2\,\hat{\mathbf{z}}} \quad \text{(same as (a))}.$$

(c) Let the charge on the sphere at time t be $q(t)$; the charge *density* is $\sigma = \frac{q(t)}{4\pi R^2}$. The charge below ("south of") the ring in the figure is

$$q_s = \sigma\left(2\pi R^2\right)\int_0^{\pi}\sin\theta'\,d\theta' = \frac{q}{2}\left(-\cos\theta'\right)\big|_\theta^\pi = \frac{q}{2}(1+\cos\theta).$$

So the total current crossing the ring (flowing "north") is $I(t) = -\frac{1}{2}\frac{dq}{dt}(1+\cos\theta)$, and hence $\mathbf{K}(t) = \frac{I}{2\pi R\sin\theta}(-\hat{\boldsymbol{\theta}}) = \frac{1}{4\pi R}\frac{dq}{dt}\frac{(1+\cos\theta)}{\sin\theta}\,\hat{\boldsymbol{\theta}}$. The force on a patch of area da is $d\mathbf{F} = (\mathbf{K}\times\mathbf{B})\,da$.

$$\mathbf{B}_{\text{ave}} = \left[\frac{2}{3}\mu_0 M\,\hat{\mathbf{z}} + \frac{\mu_0}{4\pi}\frac{\frac{4}{3}\pi R^3 M}{R^3}(2\cos\theta\,\hat{\mathbf{r}} + \sin\theta\,\hat{\boldsymbol{\theta}})\right]\frac{1}{2} = \frac{\mu_0 M}{6}[2\,\hat{\mathbf{z}} + 2\cos\theta\,\hat{\mathbf{r}} + \sin\theta\,\hat{\boldsymbol{\theta}}];$$

$$\mathbf{K}\times\mathbf{B} = \frac{1}{4\pi R}\frac{dq}{dt}\frac{\mu_0 M}{6}\frac{(1+\cos\theta)}{\sin\theta}[2(\hat{\boldsymbol{\theta}}\times\hat{\mathbf{z}}) + 2\cos\theta\,\underbrace{(\hat{\boldsymbol{\theta}}\times\hat{\mathbf{r}})}_{-\hat{\boldsymbol{\phi}}}].$$

$$\begin{aligned} d\mathbf{N} &= R\,\hat{\mathbf{r}}\times d\mathbf{F} = \frac{\mu_0 M}{24\pi}\left(\frac{dq}{dt}\right)\frac{(1+\cos\theta)}{\sin\theta}2[\underbrace{\hat{\mathbf{r}}\times(\hat{\boldsymbol{\theta}}\times\hat{\mathbf{z}})}_{\hat{\boldsymbol{\theta}}(\hat{\mathbf{r}}\cdot\hat{\mathbf{z}}) - \hat{\mathbf{z}}(\hat{\mathbf{r}}\cdot\hat{\boldsymbol{\theta}})} - \cos\theta\,\underbrace{(\hat{\mathbf{r}}\times\hat{\boldsymbol{\phi}})}_{-\hat{\boldsymbol{\theta}}}]R^2\sin\theta\,d\theta\,d\phi \\ &= \frac{\mu_0 M}{12\pi}\left(\frac{dq}{dt}\right)(1+\cos\theta)R^2[\cos\theta\,\hat{\boldsymbol{\theta}} + \cos\theta\,\hat{\boldsymbol{\theta}}]\,d\theta\,d\phi = \frac{\mu_0 MR^2}{6\pi}\left(\frac{dq}{dt}\right)(1+\cos\theta)\cos\theta\,d\theta\,d\phi\,\hat{\boldsymbol{\theta}}. \end{aligned}$$

The x and y components integrate to zero; $(\hat{\boldsymbol{\theta}})_z = -\sin\theta$, so (using $\int_0^{2\pi}d\phi = 2\pi$):

$$\begin{aligned} N_z &= -\frac{\mu_0 MR^2}{6\pi}\left(\frac{dq}{dt}\right)(2\pi)\int_0^{\pi}(1+\cos\theta)\cos\theta\sin\theta\,d\theta = -\frac{\mu_0 MR^2}{3}\left(\frac{dq}{dt}\right)\left(\frac{\sin^2\theta}{2} - \frac{\cos^3\theta}{3}\right)\Big|_0^{\pi} \\ &= -\frac{\mu_0 MR^2}{3}\left(\frac{dq}{dt}\right)\left(\frac{2}{3}\right) = -\frac{2\mu_0}{9}MR^2\frac{dq}{dt}. \quad \boxed{\mathbf{N} = -\frac{2\mu_0}{9}MR^2\frac{dq}{dt}\,\hat{\mathbf{z}}.} \end{aligned}$$

Therefore

$$\mathbf{L} = \int\mathbf{N}\,dt = -\frac{2\mu_0}{9}MR^2\,\hat{\mathbf{z}}\int_Q^0 dq = \boxed{\frac{2\mu_0}{9}MR^2Q\,\hat{\mathbf{z}}} \quad \text{(same as (a))}.$$

(I used the *average* field at the discontinuity—which is the correct thing to do—but in this case you'd get the same answer using either the inside field or the outside field.)

Question 2

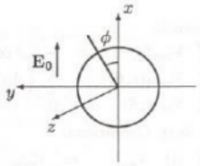
A long cylinder of linear dielectric material is placed in an otherwise uniform electric field $E_0\hat{\mathbf{z}}$. Find the resulting field inside the cylinder.

Solution

Problem 4.22

Same method as Ex. 4.7: solve Laplace's equation for $V_{\text{in}}(s,\phi)$ $(s < a)$ and $V_{\text{out}}(s,\phi)$ $(s > a)$, subject to the boundary conditions

$$\begin{cases} \text{(i)}\ V_{\text{in}} = V_{\text{out}} & \text{at } s = a, \\ \text{(ii)}\ \epsilon \frac{\partial V_{\text{in}}}{\partial s} = \epsilon_0 \frac{\partial V_{\text{out}}}{\partial s} & \text{at } s = a, \\ \text{(iii)}\ V_{\text{out}} \to -E_0 s\cos\phi & \text{for } s \gg a. \end{cases}$$



From Prob. 3.23 (invoking boundary condition (iii)):

$$V_{\text{in}}(s,\phi) = \sum_{k=1}^{\infty} s^k (a_k \cos k\phi + b_k \sin k\phi), \quad V_{\text{out}}(s,\phi) = -E_0 s\cos\phi + \sum_{k=1}^{\infty} s^{-k}(c_k \cos k\phi + d_k \sin k\phi).$$

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(I eliminated the constant terms by setting $V = 0$ on the $y\,z$ plane.) Condition (i) says

$$\sum a^k (a_k \cos k\phi + b_k \sin k\phi) = -E_0 s \cos\phi + \sum a^{-k}(c_k \cos k\phi + d_k \sin k\phi),$$

while (ii) says

$$\epsilon_r \sum k a^{k-1}(a_k \cos k\phi + b_k \sin k\phi) = -E_0 \cos\phi - \sum k a^{-k-1}(c_k \cos k\phi + d_k \sin k\phi).$$

Evidently $b_k = d_k = 0$ for all k, $a_k = c_k = 0$ unless $k = 1$, whereas for $k = 1$,

$$a a_1 = -E_0 a + a^{-1} c_1, \quad \epsilon_r a_1 = -E_0 - a^{-2} c_1.$$

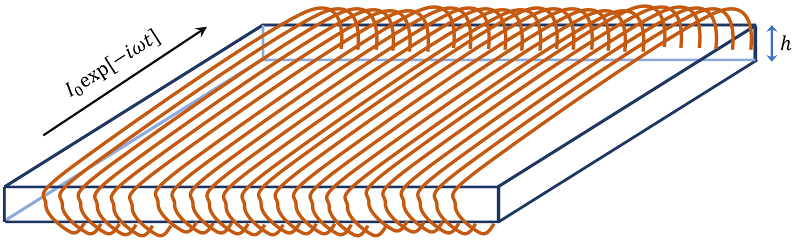
Solving for a_1,

$$a_1 = -\frac{E_0}{(1+\chi_e/2)}, \quad \text{so } V_{\text{in}}(s,\phi) = -\frac{E_0}{(1+\chi_e/2)} s\cos\phi = -\frac{E_0}{(1+\chi_e/2)} x,$$

and hence $\mathbf{E}_{\text{in}}(s,\phi) = -\frac{\partial V_{\text{in}}}{\partial x}\hat{\mathbf{x}} = \boxed{\frac{\mathbf{E}_0}{(1+\chi_e/2)}}$. As in the spherical case (Ex. 4.7), the field inside is *uniform*.

Question 3

A coil in which a current $I_0 \exp(-i\omega t)$ is flowing, is wound around a wide plate which has conductivity σ and magnetic permeability μ. The plate thickness is $2h$, such that it is bounded by the planes $x = \pm h$.



The number of turns of the coil (which are parallel to one another) per unit length is n and the thickness of the coil is very small. Neglecting edge effects and using the quasi-static approximation:

1. Determine the magnitude of the magnetic field inside the plate.
2. Use the result of (1) to determine the behavior of the field in the limiting cases of a strong ($\delta \ll h$) and weak ($\delta \gg h$) skin effect ($\delta = c/\sqrt{2\pi\mu\sigma\omega}$).

Solution

In the quasi-static approximation we have

$$\nabla^2 H = -\frac{4\pi\mu i\omega\sigma}{c} H,$$

where we've used the form $H \sim e^{i\omega t}$. Since the plate is very wide and long, there is no dependence on any coordinate besides x. The magnetic field is in the $\hat{\mathbf{z}}$ direction (similar to class 7 question 3), so ultimately we have a very simple equation to solve,

$$\frac{d^2 H_z(x)}{dx^2} = -\frac{4\pi\mu i\omega\sigma}{c} H_z(x).$$

The solution is

$$H_z(x) = A\cos((i+1)x/\delta) + B\sin((i+1)x/\delta),$$

and using the boundary conditions $H(x = \pm h) = H_0$, we obtain a pair of equations for the coefficients A, B:

$$\begin{aligned} H_0 =& A\cos\left(\left(i+1\right)h/\delta\right) + B\sin\left(\left(i+1\right)h/\delta\right) \\ H_0 =& A\cos\left(\left(i+1\right)h/\delta\right) - B\sin\left(\left(i+1\right)h/\delta\right). \end{aligned}$$

Adding and subtracting these gives

$$A = \frac{H_0}{\cos\left(\left(i+1\right)h/\delta\right)}, B = 0.$$

The solution is then

$$H_z(x) = H_0 \frac{\cos\left(\left(i+1\right)x/\delta\right)}{\cos\left(\left(i+1\right)h/\delta\right)}.$$

Outside the plate, the coil produces a field

$$H_z = H_0 = \frac{4\pi}{c} I_0 n,$$

and thus

$$H_z(x) = \frac{4\pi}{c} I_0 n \frac{\cos\left(\left(i+1\right)x/\delta\right)}{\cos\left(\left(i+1\right)h/\delta\right)}.$$

1. To obtain the magnitude, we use the exponential form of the cosine function,

$$\begin{aligned} \cos\left(\left(i+1\right)x/\delta\right) &= \frac{\exp\left(i\left(i+1\right)x/\delta\right) + \exp\left(-i\left(i+1\right)x/\delta\right)}{2} \\ &= \frac{\exp\left(\left(-1+i\right)x/\delta\right) + \exp\left(\left(1-i\right)x/\delta\right)}{2} \\ &= \frac{\exp\left(-x/\delta\right)\left[\cos\left(x/\delta\right) + i\sin\left(x/\delta\right)\right] + \exp\left(x/\delta\right)\left[\cos\left(x/\delta\right) - i\sin\left(x/\delta\right)\right]}{2} \\ &= \frac{\left(\exp\left(x/\delta\right) + \exp\left(-x/\delta\right)\right)\cos\left(x/\delta\right) + i\sin\left(x/\delta\right)\left(\exp\left(-x/\delta\right) - \exp\left(x/\delta\right)\right)}{2} \\ &= \cosh\left(x/\delta\right)\cos\left(x/\delta\right) - i\sin\left(x/\delta\right)\sinh\left(x/\delta\right). \end{aligned}$$

To simplify the computation, we notice that given some complex number

$$z = \frac{a+ib}{c+id} = \frac{\left(a+ib\right)\left(c-id\right)}{c^2+d^2},$$

its magnitude is given by

$$
\begin{aligned}
|z| &= \sqrt{\frac{(ac+bd+i(bc-ad))(ac+bd-i(bc-ad))}{(c^2+d^2)^2}} \\
&= \sqrt{\frac{(ac+bd)^2+(bc-ad)^2}{(c^2+d^2)^2}} = \left(\frac{a^2+b^2}{c^2+d^2}\right)^{1/2}.
\end{aligned}
$$

Therefore, we can complex conjugate both the denominator and numerator separately, and obtain the magnitude of the magnetic field:

$$
\begin{aligned}
|H(x)| &= H_0\sqrt{\frac{(\cosh(x/\delta)\cos(x/\delta)-i\sin(x/\delta)\sinh(x/\delta))(\cosh(x/\delta)\cos(x/\delta)+i\sin(x/\delta)\sinh(x/\delta))}{(\cosh(h/\delta)\cos(h/\delta)-i\sin(h/\delta)\sinh(h/\delta))(\cosh(h/\delta)\cos(h/\delta)+i\sin(h/\delta)\sinh(h/\delta))}} \\
&= H_0\sqrt{\frac{\cosh^2(x/\delta)\cos^2(x/\delta)+\sinh^2(x/\delta)\sin^2(x/\delta)}{\cosh^2(h/\delta)\cos^2(h/\delta)+\sinh^2(h/\delta)\sin^2(h/\delta)}} \\
&= H_0\sqrt{\frac{\left(1+\sinh^2(x/\delta)\right)\cos^2(x/\delta)+\sinh^2(x/\delta)\sin^2(x/\delta)}{\left(1+\sinh^2(h/\delta)\right)\cos^2(h/\delta)+\sinh^2(h/\delta)\sin^2(h/\delta)}} \\
&= H_0\sqrt{\frac{\cos^2(x/\delta)+\sinh^2(x/\delta)}{\cos^2(h/\delta)+\sinh^2(h/\delta)}},
\end{aligned}
$$

where we used the identities $\cosh^2(x)-\sinh^2(x)=1=\cos^2(x)+\sin^2(x)$. The real part of the amplitude is therefore

$$
|H(x)| = \frac{4\pi}{c}I_0 n\left[\frac{\cos^2\left(xc/\sqrt{2\pi\mu\sigma\omega}\right)+\sinh^2\left(xc/\sqrt{2\pi\mu\sigma\omega}\right)}{\cos^2\left(hc/\sqrt{2\pi\mu\sigma\omega}\right)+\sinh^2\left(hc/\sqrt{2\pi\mu\sigma\omega}\right)}\right]^{1/2}.
$$

2. For strong skin effects $(h/\delta \gg 1)$ we have

$$
\begin{aligned}
|H(x)| &= H_0\sqrt{\frac{\cos^2(x/\delta)+\sinh^2(x/\delta)}{\underbrace{\cos^2(h/\delta)}_{\text{bounded by }\pm1}+\sinh^2(h/\delta)}} \\
&\approx H_0\frac{|\sinh(x/\delta)|}{\exp(h/\delta)} \\
&\approx \frac{H_0}{2}\exp(-h/\delta)\left(\exp(|x|/\delta)-\underbrace{\exp(-|x|/\delta)}_{\ll\exp(|x|/\delta)}\right) \\
&\approx \frac{H_0}{2}\exp\left(-(h-|x|)/\delta\right),
\end{aligned}
$$

and since $|x| < h$ we get strong exponential decay of the field beyond the depth $\sim \delta$.

In the weak skin effect limit ($h/\delta \ll 1$) we find

$$\begin{aligned} |H(x)| &= H_0 \sqrt{\frac{\cos^2(x/\delta) + \overbrace{\sinh^2(x/\delta)}^{\approx 0}}{\cos^2(h/\delta) + \underbrace{\sinh^2(h/\delta)}_{\approx 0}}} \\ &\approx H_0 \frac{|\cos(x/\delta)|}{|\cos(h/\delta)|}, \end{aligned}$$

and since compared to δ, x is of the same order as h, we have

$$|H(x)| \approx H_0,$$

which means the magnetic field completely penetrates the conductor, which is reasonable, given that the skin-depth δ is very large compared to the system size.