Class Exercise 8 - Electromagnetic waves in matter

Problem 1

Consider a wave with frequency ω, polarized in the plane of incidence, which propagates through three different media with refraction indices n_1, n_2 and n_3. The width of the second medium with n_2 is d. The wave hits this medium parallel to its normal.

1. Find the transmission coefficient.
2. Can there be pure reflection of the wave?
3. For $\mu_1 = \mu_2 = \mu_3 = n_3 = 1$, can there be pure transmission of the wave? What thickness d and refraction index n_2 are necessary in that case?

Solution

1. The transmission coefficient is related to the Poynting vector on each side through

$$T = \frac{S_T}{S_I} = \frac{\frac{c}{4\pi\mu_3}\left|\vec{E}_T \times \vec{B}_T\right|}{\frac{c}{4\pi\mu_1}\left|\vec{E}_I \times \vec{B}_I\right|} = \frac{\mu_1\left|\vec{E}_T \times \vec{B}_T\right|}{\mu_3\left|\vec{E}_I \times \vec{B}_I\right|} \underset{E\perp B}{=} \frac{\mu_1\left|\vec{E}_T\right|\left|\vec{B}_T\right|}{\mu_3\left|\vec{E}_I\right|\left|\vec{B}_I\right|}.$$

Since $\vec{E}$ and $\vec{k}$ are given, we can compute $\vec{B}$,

$$\vec{\nabla} \times \vec{E} = -\frac{1}{c}\frac{\partial \vec{B}}{\partial t} \implies i\vec{k} \times \vec{E} = \frac{i\omega}{c}\vec{B},$$

where

$$\omega = kv = kc/n,$$

such that

$$\vec{B} = n\hat{\mathbf{k}} \times \vec{E}.$$

We therefore find that the transmission coefficient

$$T = \frac{\mu_1 n_3}{\mu_3 n_1} \frac{E_T^2}{E_I^2}.$$

We choose the plane of incidence to be $z = 0$, and thus

$$\begin{aligned}
\vec{E}_I &= E_I e^{i(k_I z - \omega t)} \hat{\mathbf{x}}, \\
\vec{B}_I &= n_1 E_I e^{i(k_I z - \omega t)} \hat{\mathbf{y}}, \\
\vec{E}_R &= E_R e^{i(k_R z - \omega t)} \hat{\mathbf{x}}, \\
\vec{B}_R &= -n_1 E_R e^{i(k_R z - \omega t)} \hat{\mathbf{y}}, \\
\vec{E}_1 &= E_1 e^{i(k_1 z - \omega t)} \hat{\mathbf{x}}, \\
\vec{B}_1 &= n_2 E_1 e^{i(k_1 z - \omega t)} \hat{\mathbf{y}}, \\
\vec{E}_2 &= E_1 e^{i(k_2 z - \omega t)} \hat{\mathbf{x}}, \\
\vec{B}_2 &= -n_2 E_1 e^{i(k_2 z - \omega t)} \hat{\mathbf{y}}, \\
\vec{E}_T &= E_T e^{i(k_T z - \omega t)} \hat{\mathbf{x}}, \\
\vec{B}_T &= n_3 E_T e^{i(k_T z - \omega t)} \hat{\mathbf{x}}.
\end{aligned}$$

The boundary conditions are

$$\begin{aligned}
\frac{B_\parallel^-}{\mu^-} &= \frac{B_\parallel^+}{\mu^+}, \\
E_\parallel^- &= E_\parallel^+,
\end{aligned}$$

and we have

$$E_\perp = E_z = B_\perp = B_z = 0,$$

so no need to concern ourselves with the perpendicular boundary conditions. Using the parallel boundary conditions for $\vec{E}$ we find

$$z = 0: \quad E_I + E_R = E_1 + E_2, \tag{1}$$

$$z = d: \quad E_1 \exp\left[i\frac{\omega d n_2}{c}\right] + E_2 \exp\left[-i\frac{\omega d n_2}{c}\right] = E_T \exp\left[i\frac{\omega d n_3}{c}\right], \tag{2}$$

where in the second condition we used $k_I = -k_R$. The conditions for $\vec{B}$ give

$$z = 0: \quad \frac{n_1}{\mu_1}\left(E_I - E_R\right) = \frac{n_2}{\mu_2}\left(E_1 - E_2\right), \tag{3}$$

$$z = d: \quad \frac{n_2}{\mu_2}\left(E_1 \exp\left[i\frac{\omega d n_2}{c}\right] - E_2 \exp\left[-i\frac{\omega d n_2}{c}\right]\right) = \frac{n_3}{\mu_3} E_T \exp\left[i\frac{\omega d n_3}{c}\right]. \tag{4}$$

To we add (1) and $(\mu_1/n_1)\,(3)$ to eliminate E_R and obtain

$$2E_I = E_1\left(1+\frac{\mu_1 n_2}{\mu_2 n_1}\right)+\left(1-\frac{\mu_1 n_2}{\mu_2 n_1}\right)E_2.$$

Adding and subtracting (2), $(\mu_2/n_2)\,(4)$, we find

$$E_1 = \frac{1}{2}\left(1+\frac{\mu_2 n_3}{\mu_3 n_2}\right)E_T\exp\left[i\frac{\omega d\,(n_3-n_2)}{c}\right],$$
$$E_2 = \frac{1}{2}\left(1-\frac{\mu_2 n_3}{\mu_3 n_2}\right)E_T\exp\left[i\frac{\omega d\,(n_3+n_2)}{c}\right].$$

Combining the expressions of E_I and E_T, we obtain

$$\frac{E_T}{E_I} = \frac{2\mu_3 n_2 \mu_2 n_1 \exp\left[i\omega d n_3/c\right]}{\left(\mu_3 n_2\mu_2 n_1+\mu_2 n_3\mu_1 n_2\right)\cos\left(\omega d n_2/c\right)-i\left(\mu_3\mu_1 n_2^2+\mu_2^2 n_3 n_1\right)\sin\left(\omega d n_2/c\right)}.$$

The transmission coefficient is then

$$T = \frac{\mu_3 n_3 \mu_1 n_1\left(2n_2\mu_2\right)^2}{\left(\mu_3 n_2\mu_2 n_1+\mu_2 n_3\mu_1 n_2\right)^2\cos^2\left(\omega d n_2/c\right)+\left(\mu_3\mu_1 n_2^2+\mu_2^2 n_3 n_1\right)^2\sin^2\left(\omega d n_2/c\right)}.$$

2. In order to have full reflection ($T=0$) with real valued refraction indices, we will need to have either $n_2\to\infty$ or $n_3\to\infty$, which is obviously impossible to achieve. You saw in class that conductors, which make excellent mirrors, have complex valued refraction indices! The imaginary part accounts for the dissipation of energy into heat inside the material.

3. The reflection coefficient is

$$R = \left|\frac{\left(1-\frac{n_2}{n_1}\right)\left(1+\frac{1}{n_2}\right)+\left(1+\frac{n_2}{n_1}\right)\left(1-\frac{1}{n_2}\right)e^{i\omega d n_2/c}}{\left(1+\frac{n_2}{n_1}\right)\left(1+\frac{1}{n_2}\right)+\left(1-\frac{n_2}{n_1}\right)\left(1-\frac{1}{n_2}\right)e^{i\omega d n_2/c}}\right|^2,$$

which means its minimal value is obtained when $e^{i\omega d n_2/c}=-1$, which means

$$d=\frac{\pi c}{\omega n_2}.$$

In that case, it remains to be found when

$$\left(1-\frac{n_2}{n_1}\right)\left(1+\frac{1}{n_2}\right)=\left(1+\frac{n_2}{n_1}\right)\left(1-\frac{1}{n_2}\right),$$

which boils down to a simple quadratic equation

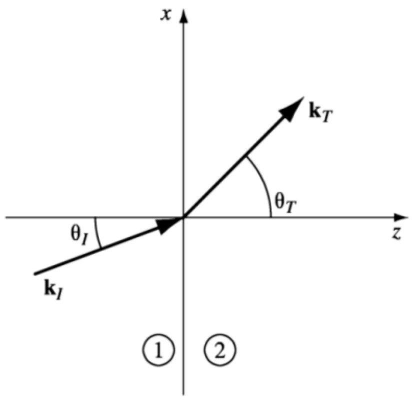
$$0 = n_2^2 + n_2 - n_1,$$

with the solution

$$n_2 = \frac{-1 + \sqrt{1 + 4n_1}}{2}.$$

Problem 2 - Evanescent waves (total internal reflection)

An EM wave polarized parallel to the plane of incidence with frequency ω travels between materials with refraction indices $n_1 > n_2$ and $\mu_1 = \mu_2$.



1. Find the critical angles $\theta_c^{(T)}$ beyond which (namely, for $\theta_I > \theta_c^{(T)}$ there is no transmission. Find the Brewster angle $\theta_c^{(R)}$ for which there is no reflection.
2. Is $\theta_c^{(R)}$ bigger or smaller than $\theta_c^{(T)}$? Is it possible to find $\theta_c^{(R)}$ for $\theta_I > \theta_c^{(T)}$?
3. For $\theta_I > \theta_c^{(T)}$ consider
$$\vec{k}_T = \frac{\omega n_2}{c}\left(\sin\theta_T \hat{\mathbf{x}} + \cos\theta_T \hat{\mathbf{z}}\right).$$
Show that the fields $\vec{E}, \vec{B}$ decay exponentially (*evanescent waves*) in the region (2) .
4. Find the energy flow across $z = 0$ in the region $z > 0$.

Solution

1. According to Snell's law we have
$$\frac{\sin\theta_T}{\sin\theta_I} = \frac{n_1}{n_2},$$
so the transmission of the waves is exactly zero when $\theta_T = \pi/2$, namely when
$$\sin\theta_I = n_2/n_1 = \sin\theta_c^{(T)}.$$
Therefore
$$\theta_c^{(T)} = \arcsin\left(n_2/n_1\right).$$
To obtain the critical angle for reflection, recall the Fresnel equations you saw in class,
$$\frac{E_R}{E_I} = \frac{\alpha - \beta}{\alpha + \beta},$$

where

$$\alpha = \frac{\cos\theta_T}{\cos\theta_I}, \beta = \frac{n_2}{n_1} \text{ (for } \mu_1 = \mu_2).$$

The reflection vanishes when $\alpha = \beta$, or

$$\frac{\cos\theta_T}{\cos\theta_c^{(R)}} = \frac{n_2}{n_1}.$$

We use Snell's law once again to write

$$\frac{n_2}{n_1} = \frac{\cos\theta_T}{\cos\theta_I} = \frac{\sqrt{1-\sin^2\theta_T}}{\cos\theta_I} = \frac{\sqrt{1-\left(\frac{n_1}{n_2}\right)^2 \sin^2\theta_c^{(R)}}}{\cos\theta_I},$$

and square the result to obtain the relation

$$\beta^2 = \frac{1-\beta^{-2}\sin^2\theta_c^{(R)}}{1-\sin^2\theta_c^{(R)}} \implies \sin^2\theta_c^{(R)} = \beta^2\left(\frac{1-\beta^2}{1-\beta^4}\right),$$

and therefore also

$$\cos^2\theta_c^{(R)} = \frac{1-\beta^2}{1-\beta^4},$$

which ultimately gives

$$\theta_c^{(R)} = \arctan\beta = \arctan\frac{n_2}{n_1}.$$

2. We find that since $\arcsin x > \arctan x$,

$$\theta_c^{(T)} > \theta_c^{(R)}.$$

This result is expected from conservation of energy - if for values of θ_I where transmission is 0 there could also be zero reflection, where would the energy of the incident wave disappear?

3. We get a transmitted wave

$$\begin{aligned}
\vec{E}_T &= \widetilde{E_T}\exp\left(i\vec{k}_T\cdot\vec{r} - i\omega t\right)\hat{\epsilon} \\
&= \widetilde{E_T}\exp\left(i\frac{\omega n_2}{c}\left(\sin\theta_T\hat{\mathbf{x}} + \cos\theta_T\hat{\mathbf{z}}\right)\cdot\vec{r} - i\omega t\right)\hat{\epsilon} \\
&= \widetilde{E_T}\exp\left(i\frac{\omega n_2}{c}\left(\sin\theta_T x + \cos\theta_T z\right) - i\omega t\right)\hat{\epsilon}.
\end{aligned}$$

Notice that for $\theta_I > \theta_c^{(T)}$ we find an odd result: since

$$\sin\theta_I > \sin\theta_c^{(T)} = \frac{n_2}{n_1},$$

we have from Snell's law that

$$\sin\theta_T = \frac{n_1}{n_2}\sin\theta_I > \frac{n_1}{n_2}\frac{n_2}{n_1} = 1.$$

How is that possible? This is not a mistake - it just means θ_T must be complex! Consequently,

$$\cos\theta_T = \sqrt{1-\sin^2\theta_T} = i\underbrace{\sqrt{\sin^2\theta_T - 1}}_{>0},$$

which means $\mathfrak{Im}\cos\theta_T > 0$ and since it is purely imaginary, the wave has a decaying piece proportional to $\exp\left(-\omega n_2 \left|\cos\theta_T\right| z/c\right)$ and a decay in the magnetic field as well, since $\vec{k}\times\vec{E}\propto\vec{B}$.

4. The energy flux for complex fields is given by (HW8)

$$S = \frac{c}{4\pi}\mathfrak{Re}\left(\vec{E}\times\vec{B^*}\right).$$

In the $\hat{\mathbf{z}}$ direction and $z > 0$, we find

$$\begin{aligned} S_z &= \frac{c}{4\pi}\mathfrak{Re}\left(\vec{E}\times\vec{B^*}\right)_z \\ &= \frac{c}{4\pi}\mathfrak{Re}\left(\vec{E}_T\times\left(n_2\vec{k_T}\times\vec{E}_T\right)^*\right)_z \\ &= \frac{n_2 c}{4\pi}\mathfrak{Re}\left(\hat{\mathbf{z}}\cdot\vec{k_T}\left|\vec{E}_T\right|^2\right). \end{aligned}$$

We note that

$$\hat{\mathbf{z}}\cdot\vec{k_T} = k\cos\theta_T,$$

and that $\cos\theta_T$ is purely imaginary, which means

$$S_z = 0.$$

We get the physically expected result - that the energy flux in region (2) is identically zero. Energy does not transfer to that region, nor does it decay exponentially within it, in consistency with the condition that $\theta_c^{(T)}$ is the critical angle for energy transfer.

Problem 3

Find the electromagnetic field of a TM wave in a cylindrical waveguide of radius a.

Solution

We take the axis of symmetry to be $\hat{\mathbf{z}}$, and therefore the TM wave will have $B_z = 0$. We note that it suffices to determine only the E_z component of the electric field, since $E_r, E_\varphi, B_r, B_\varphi$ can be derived from it using Maxwell's equations.

The wave equation follows from

$$\nabla^2 E_z = \frac{1}{c^2}\frac{\partial^2 E_z}{\partial t^2},$$

and we assume E_z has the separable form

$$E_z = R(r)\,\Phi(\varphi)\,e^{i(kz-\omega t)},$$

where we used the fact that the cylinder is a waveguide to determine that the only direction of propagation we are interested in is $\hat{\mathbf{z}}$. Thus, in cylindrical coordinates we solve the equation

$$\begin{aligned} \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial E_z}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 E_z}{\partial \varphi^2} + \frac{\partial^2 E_z}{\partial z^2} &= \frac{1}{c^2}\frac{\partial^2 E_z}{\partial t^2}, \\ \implies \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial E_z}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 E_z}{\partial \varphi^2} + \left(\frac{\omega^2}{c^2} - k^2\right) E_z &= 0. \end{aligned}$$

We substitute $\Phi(\varphi) \propto e^{\pm im\varphi}$ and obtain the Bessel equation

$$\frac{d^2R}{dr^2} + \frac{1}{r}\frac{dR}{dr} + \left(\frac{\omega^2}{c^2} - k^2 - \frac{m^2}{r^2}\right) R = 0,$$

with the solution being a Bessel function of the first kind (the second kind diverges at $r \to 0$),

$$R_m(r) \propto J_m\left(\sqrt{\frac{\omega^2}{c^2} - k^2 r}\right).$$

The boundary condition on the waveguide is that $E_\parallel(r = a) = 0$, which gives the condition

$$J_m\left(\sqrt{\frac{\omega^2}{c^2} - k^2 a}\right) = 0 \implies \sqrt{\frac{\omega^2}{c^2} - k^2 a} = \chi_{n,m},$$

where $\chi_{n,m}$ is the n-th root of the order n Bessel function and they satisfy $\chi_{n,m} < \chi_{n+1,m}$.

Inverting this relation gives

$$\omega_{n,m} = c\sqrt{k^2 + \frac{\chi_{n,m}^2}{a^2}},$$

and we find that waves with frequencies $\omega < (c/a)\,\chi_{n,m} \equiv \omega_{n,m}$ have an imaginary k which causes exponential decay of the wave. The frequency $\omega_{n,m}$ is called the *cutoff frequency*, and below this value the waveguide will exponentially attenuate the field. The lowest cutoff frequency is then $\omega_{1,m}$, and waves which have frequencies less than this will not propagate at all through the guide.

We find the field

$$E_z\,(r,\varphi,z,t)_{mn} = E_0 e^{i(kz-\omega t+m\varphi)} J_m\left(\chi_{m,n}\frac{r}{a}\right).$$

This mode of the field is called the TM_{mn} mode. Combining the Maxwell equations

$$\vec{\nabla}\times\vec{E} = -\frac{1}{c}\frac{\partial\vec{B}}{\partial t}, \vec{\nabla}\times\vec{B} = \frac{\mu\epsilon}{c}\frac{\partial\vec{E}}{\partial t}$$

will give us the rest of the component in terms of E_z, ω, k (home).