Homework 7

Question 1

Consider a couple of real functions

$$f\left(t,\vec{r}\right)=A\cos\left(\vec{k}\cdot\vec{r}-\omega t+\delta_a\right),$$
$$g\left(t,\vec{r}\right)=B\cos\left(\vec{k}\cdot\vec{r}-\omega t+\delta_b\right).$$

Show that their average product is equal to the real part of the product of their complex couterparts $\widetilde{f},\widetilde{g}$, such that

$$\langle fg\rangle=\frac{1}{2}\mathfrak{Re}\left(\widetilde{f}\widetilde{g}^{*}\right),$$

where the average is taken over time,

$$\langle f\rangle=\frac{1}{T}\int_0^T f\,\mathrm{d}t.$$

Remark This proves the relation of the energy and Poynting vector to the complex fields we deal with in calculations of electromagnetic waves. For example,

$$\langle u\rangle=\frac{1}{8\pi}\mathfrak{Re}\left(\epsilon\vec{\widetilde{E}}\cdot\vec{\widetilde{E}}^{*}+\frac{1}{\mu}\vec{\widetilde{B}}\cdot\vec{\widetilde{B}}^{*}\right),\left\langle\vec{S}\right\rangle=\frac{c}{4\pi}\mathfrak{Re}\left(\vec{\widetilde{E}}\times\vec{\widetilde{B}}^{*}\right).$$

Question 2

Spherical wave
Suppose

$$\vec{E}(r,\theta,\phi,t) = A\frac{sin\theta}{r}\left[cos(kr-\omega t) - \frac{1}{kr}sin(kr-\omega t)\right]\hat{\phi} \tag{1}$$

with $\frac{\omega}{k} = -c$. (This is, incidentally, the simplest possible spherical wave. For notational convenience, let $(kr - \omega t) = u$ in your calculations.)

(a) Show that $\vec{E}$ obeys all four of Maxwell's equations, in vacuum, and find the associated magnetic field.

(b) Calculate the Poynting vector. Average $\vec{S}$ over a full cycle to get the intensity vector $\vec{I}$. (Does it point in the expected direction? Does it fall off like r^2 , as it should?)

(c) Integrate $\vec{I} \cdot d\vec{a}$ over a spherical surface to determine the total power radiated.

Question 3

A monochromatic plane wave of frequency ω is incident normally on a stack of layers of various thicknesses t_j and indices of refraction n_j. Inside the stack, the has both forward and backward moving components. The change in wave through any interface and also from one side of a layer to the other can be described by means of 2×2 transfer matrices. If the electric field is written as

$$E = E_+ e^{ikx} + E_- e^{-ikx}$$

in each layer, the transfer matrix equation $E' = TE$ is explicitly

$$\begin{pmatrix} E'_+ \\ E'_- \end{pmatrix} = \begin{pmatrix} t_{11} & t_{12} \\ t_{21} & t_{22} \end{pmatrix} \begin{pmatrix} E_+ \\ E_- \end{pmatrix}.$$

1. Show that the transfer matric for propagation inside, but across, a layer of index of refraction n_j and thickness t_j is

$$T_{\text{layer}}(n_j, t_j) = \begin{pmatrix} e^{ik_j t_j} & 0 \\ 0 & e^{-ik_j t_j} \end{pmatrix} = I\cos(k_j t_j) + i\sigma_3 \sin(k_j t_j),$$

where $k_j = n_j \omega / c$, I is the unit matrix, and σ_k are the Pauli spin matrices of quantum mechanics, given by

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

2. Show that the transfer matrix to cross an interface from n_1 $(x < x_0)$ to n_2 $(x > x_0)$ is

$$T_{\text{interface}}(2,1) = \frac{1}{2}\begin{pmatrix} n+1 & -(n-1) \\ -(n-1) & n+1 \end{pmatrix} = I\frac{(n+1)}{2} - \sigma_1\frac{(n-1)}{2},$$

   where $n = n_1/n_2$.

3. Show that for a complete stack, the incident, reflected and transmitted waves are related by

$$E_T = \frac{\det(T)}{t_{22}}E_I, \quad E_R = -\frac{t_{21}}{t_{22}}E_I,$$

   where t_{ij} are the elements of the matrix T, which is the product of the forward-going transfer matrices, including from the material filled space on the incident side into the first layer, and from the last layer into the medium filling the space on the transmitted side.

Question 4

Two plane semi-infinite slabs of the same uniform, isotropic dielectric with index of refraction n, permeability $\mu = 1$ and zero conductivity, are parallel and separated by an air gap $(n = 1)$ of width d. A plane electromagnetic wave of frequency ω is incident on the gap from one of the slabs with an angle of incidence θ_I. For linear polarization both parallel and perpendicular to the plane of incidence,

1. calculate the ratio of power transmitted into the second slab to the incident power, as well as the ratio of reflected to incident power,

2. for θ_I greater than the critical angle for total internal reflection $\left(\theta_c^{(T)} \text{ from class 8}\right)$, sketch (roughly) the ratio of transmitted power to incident power as a function of d, measured in units of wavelength in the gap.