Class Exercise 9 - Retarded potentials

Problem 1

An infinite wire directed along $\hat{\mathbf{z}}$ carries a current

$$I(t) = \begin{cases} 0 & t < 0 \\ kt & t > 0 \end{cases}$$

Find the electromagnetic fields.

Solution

The current enters only in the vector potential,

$$\vec{A} = \frac{1}{c} \iiint \frac{\vec{J}(\vec{r}', t_r)}{|\vec{r} - \vec{r}'|} \mathrm{d}^3 r',$$

where $\vec{J} \mathrm{d}^3 r' = I \,\mathrm{d}z \hat{\mathbf{z}}$. The current in the retarded time $t_{\mathrm{ret}} = t - |\vec{r} - \vec{r}'| / c$ is then given by

$$I(t_{\mathrm{ret}}) = \begin{cases} 0 & t_{\mathrm{ret}} < 0 \\ k(t - |\vec{r} - \vec{r}'| / c) & t_{\mathrm{ret}} > 0 \end{cases}$$

where $t_{\mathrm{ret}} > 0$ means $ct > |\vec{r} - \vec{r}'|$ - a time greater than the time it takes the information of the field to arrive from point $\vec{r}'$ to point $\vec{r}$, signifying the earliest time a point in spacetime (t, r) will know that the current has been switched on. The current flows in the $\hat{\mathbf{z}}$ direction so we only need to find A_z. The integration is done over z', so we want to express the current in terms of z' using the condition that $t_{\mathrm{ret}} > 0$. We square the condition and obtain

$$(ct)^2 > (\vec{r} - \vec{r}')^2,$$

and use

$$\vec{r'} = z'\hat{\mathbf{z}},$$
$$\vec{r} = x\hat{\mathbf{x}} + y\hat{\mathbf{y}} + z\hat{\mathbf{z}},$$

to rewrite the condition explicitly in Cartesian coordinates as

$$(ct)^2 > x^2 + y^2 + (z - z')^2 .$$

Rearranging this expression and taking its square-root gives a condition on the range of values z' spans,

$$\sqrt{(ct)^2 - (x^2 + y^2)} > |z - z'|$$
$$\implies z - \sqrt{(ct)^2 - (x^2 + y^2)} < z' < z + \sqrt{(ct)^2 - (x^2 + y^2)}.$$

The integral for A becomes

$$\begin{aligned}\vec{A} &= \frac{1}{c^2} \int\limits_{z-\sqrt{(ct)^2-(x^2+y^2)}}^{z+\sqrt{(ct)^2-(x^2+y^2)}} \frac{k\left(ct - \sqrt{x^2 + y^2 + (z - z')^2}\right)}{\sqrt{x^2 + y^2 + (z - z')^2}} \,\mathrm{d}z'\hat{\mathbf{z}} \\ &= \hat{\mathbf{z}}\frac{k}{c^2}\left[\int\limits_{z-\sqrt{(ct)^2-(x^2+y^2)}}^{z+\sqrt{(ct)^2-(x^2+y^2)}} \frac{ct\,\mathrm{d}z'}{\sqrt{x^2 + y^2 + (z - z')^2}} - 2\sqrt{(ct)^2 - (x^2 + y^2)}\right] \\ &= \hat{\mathbf{z}}\frac{k}{c^2}\left[ct \ln\left(\frac{ct + \sqrt{(ct)^2 - (x^2 + y^2)}}{ct - \sqrt{(ct)^2 - (x^2 + y^2)}}\right) - 2\sqrt{(ct)^2 - (x^2 + y^2)}\right].\end{aligned}$$

In cylindrical coordinates where $\rho = \sqrt{x^2 + y^2}$ we find

$$\vec{A} = \hat{\mathbf{z}}\frac{k}{c^2}\left[ct \ln\left(\frac{ct + \sqrt{(ct)^2 - \rho^2}}{ct - \sqrt{(ct)^2 - \rho^2}}\right) - 2\sqrt{(ct)^2 - \rho^2}\right].$$

To compute $\vec{E}, \vec{B}$ we will just need to use (home)

$$\vec{E} = -\frac{1}{c}\frac{\partial \vec{A}}{\partial t}, \vec{B} = -\frac{\partial A_z}{\partial \rho}\hat{\boldsymbol{\varphi}}.$$

Problem 2

A point charge is moving in a circular orbit of radius d. At time $t=0$ the charge is at rest at the point $(d,0)$ on the x axis. The charge is moving at constant angular velocity ω.

1. Find the Lienard-Wiechert potentials on the z axis.

2. Find the electromagnetic fields at the center of the circular orbit.

Solution

1. The location of the charge is

$$\vec{r_0}\left(t\right)=d\left[\cos\left(\omega t\right)\hat{\mathbf{x}}+\sin\left(\omega t\right)\hat{\mathbf{y}}\right].$$

We are interested in finding the potentials at $\vec{r}=z\hat{\mathbf{z}}$, where

$$\vec{R}\left(t_{\text{ret}}\right)=\vec{r}-\vec{r_0}\left(t_{\text{ret}}\right)=z\hat{\mathbf{z}}-d\left[\cos\left(\omega t_{\text{ret}}\right)\hat{\mathbf{x}}+\sin\left(\omega t_{\text{ret}}\right)\hat{\mathbf{y}}\right],$$

such that

$$R\left(t_{\text{ret}}\right)=\left|R\left(t_{\text{ret}}\right)\right|=\sqrt{z^2+d^2}.$$

To find the potentials we need to find the velocity as well,

$$\vec{v}\left(t_{\text{ret}}\right)=\frac{\mathrm{d}\vec{r_0}\left(t_{\text{ret}}\right)}{\mathrm{d}t_{\text{ret}}}=d\omega\left[-\sin\left(\omega t_{\text{ret}}\right)\hat{\mathbf{x}}+\cos\left(\omega t_{\text{ret}}\right)\hat{\mathbf{y}}\right].$$

We therefore have

$$\begin{aligned}\vec{R}\left(t_{\text{ret}}\right)\cdot\vec{v}\left(t_{\text{ret}}\right)&=\left(z\hat{\mathbf{z}}-d\left[\cos\left(\omega t_{\text{ret}}\right)\hat{\mathbf{x}}+\sin\left(\omega t_{\text{ret}}\right)\hat{\mathbf{y}}\right]\right)\cdot\left(d\omega\left[-\sin\left(\omega t_{\text{ret}}\right)\hat{\mathbf{x}}+\cos\left(\omega t_{\text{ret}}\right)\hat{\mathbf{y}}\right]\right)\\&=-d^2\omega\left[-\sin\left(\omega t_{\text{ret}}\right)\cos\left(\omega t_{\text{ret}}\right)+\sin\left(\omega t_{\text{ret}}\right)\cos\left(\omega t_{\text{ret}}\right)\right]\\&=0.\end{aligned}$$

The potentials are given by

$$\begin{aligned}\Phi\left(t,\vec{r}\right)&=\frac{qc}{cR\left(t_{\text{ret}}\right)-\vec{R}\left(t_{\text{ret}}\right)\cdot\vec{v}\left(t_{\text{ret}}\right)}=\frac{q}{\sqrt{z^2+d^2}},\\\vec{A}\left(t,\vec{r}\right)&=\frac{q\vec{v}\left(t_{\text{ret}}\right)}{cR\left(t_{\text{ret}}\right)-\vec{R}\left(t_{\text{ret}}\right)\cdot\vec{v}\left(t_{\text{ret}}\right)}=\frac{qd\omega}{\sqrt{z^2+d^2}}\left[-\sin\left(\omega t_{\text{ret}}\right)\hat{\mathbf{x}}+\cos\left(\omega t_{\text{ret}}\right)\hat{\mathbf{y}}\right].\end{aligned}$$

We substitute $t_{\text{ret}} = t - R\left(t_{\text{ret}}\right)/c = t - \sqrt{z^2+d^2}/c$ and obtain the potentials

$$\begin{aligned}\Phi\left(t,\vec{r}\right) &= \frac{q}{\sqrt{z^2+d^2}},\\ \vec{A}\left(t,\vec{r}\right) &= \frac{qd\omega}{\sqrt{z^2+d^2}}\left[-\sin\left(\omega\left(t-\frac{\sqrt{z^2+d^2}}{c}\right)\right)\hat{\mathbf{x}} + \cos\left(\omega\left(t-\frac{\sqrt{z^2+d^2}}{c}\right)\right)\hat{\mathbf{y}}\right].\end{aligned}$$

We use the equations derived in class, where the fields take the form

$$\begin{aligned}\vec{E} &= \frac{qR}{\left(\vec{R}\cdot u\right)^3}\left[\left(c^2-v^2\right)\vec{u} + \vec{R}\times\left(\vec{u}\times\vec{a}\right)\right],\\ \vec{B} &= \hat{\mathbf{R}}\times\vec{E},\end{aligned}$$

where we defined $\vec{u} = c\hat{\mathbf{R}} - \vec{v}, \vec{a} = \dot{\vec{v}}$. For $z = 0$, we have

$$\begin{aligned}R &= d,\\ \hat{\mathbf{R}} = \frac{\vec{R}}{R} &= -\frac{d\left[\cos\left(\omega t_{\text{ret}}\right)\hat{\mathbf{x}} + \sin\left(\omega t_{\text{ret}}\right)\hat{\mathbf{y}}\right]}{d}\\ &= -\left[\cos\left(\omega t_{\text{ret}}\right)\hat{\mathbf{x}} + \sin\left(\omega t_{\text{ret}}\right)\hat{\mathbf{y}}\right],\\ \vec{u} = c\hat{\mathbf{R}} - \vec{v} &= -c\left[\cos\left(\omega t_{\text{ret}}\right)\hat{\mathbf{x}} + \sin\left(\omega t_{\text{ret}}\right)\hat{\mathbf{y}}\right] - d\omega\left[-\sin\left(\omega t_{\text{ret}}\right)\hat{\mathbf{x}} + \cos\left(\omega t_{\text{ret}}\right)\hat{\mathbf{y}}\right]\\ &= -\left[c\cos\left(\omega t_{\text{ret}}\right) - d\omega\sin\left(\omega t_{\text{ret}}\right)\right]\hat{\mathbf{x}} - \left[c\sin\left(\omega t_{\text{ret}}\right) + d\omega\cos\left(\omega t_{\text{ret}}\right)\right]\hat{\mathbf{y}},\\ \vec{a} = \dot{\vec{v}} &= d\omega\frac{\mathrm{d}}{\mathrm{d}t}\left[-\sin\left(\omega t_{\text{ret}}\right)\hat{\mathbf{x}} + \cos\left(\omega t_{\text{ret}}\right)\hat{\mathbf{y}}\right]\\ &= -d\omega^2\left[\cos\left(\omega t_{\text{ret}}\right)\hat{\mathbf{x}}\sin\left(\omega t_{\text{ret}}\right)\hat{\mathbf{y}}\right]\\ &= -\omega^2\vec{r}_0\left(t_{\text{ret}}\right).\end{aligned}$$

We compute the relevant quantities,

$$\begin{aligned}\vec{R}\cdot u &= d\left[c\left(\cos^2\left(\omega t_{\text{ret}}\right) + \sin^2\left(\omega t_{\text{ret}}\right)\right)\right]\\ &= dc,\\ \vec{R}\times\left(\vec{u}\times\vec{a}\right) &= \vec{u}\left(\vec{R}\cdot\vec{a}\right) - \vec{a}\left(\vec{R}\cdot\vec{u}\right)\\ &= \vec{u}\left(-\vec{r}_0\left(t_{\text{ret}}\right)\cdot\left(-\omega^2\vec{r}_0\left(t_{\text{ret}}\right)\right)\right) - \vec{a}\left(dc\right)\\ &= \omega^2 d^2\vec{u} - dc\vec{a}.\end{aligned}$$

Therefore, we obtain

$$\begin{aligned}
\vec{E} &= \frac{qd}{(dc)^3}\left[\left(c^2-\upsilon^2\right)\vec{u}+\omega^2 d^2\vec{u}-dc\vec{a}\right] \\
&= \frac{qd}{(dc)^3}\left[\left(c^2-(d\omega)^2+\omega^2 d^2\right)\vec{u}-dc\vec{a}\right] \\
&= \frac{q}{d^2c^2}\left[c\vec{u}-d\vec{a}\right] \\
&= \frac{q}{d^2c^2}\left\{\left[\left(\omega^2 d^2-c^2\right)\cos\left(\omega t_{\mathrm{ret}}\right)+dc\omega\sin\left(\omega t_{\mathrm{ret}}\right)\right]\hat{\mathbf{x}}\right. \\
&+ \left.\left[\left(\omega^2 d^2-c^2\right)\sin\left(\omega t_{\mathrm{ret}}\right)-dc\omega\cos\left(\omega t_{\mathrm{ret}}\right)\right]\hat{\mathbf{y}}\right\},
\end{aligned}$$

and

$$\vec{B}=\hat{\mathbf{R}}\times\vec{E}=\left(\hat{R}_x\vec{E}_y-\hat{R}_y\vec{E}_x\right)\hat{\mathbf{z}}=\frac{q\omega}{dc}\hat{\mathbf{z}}.$$

Problem 3

Find the approximation of the electromagnetic field in the "far-field" region where

$$r \gg r', \frac{c}{\omega},$$

for a harmonic source of charges with frequency ω.

Solution

For a harmonic source, $\rho, J \propto e^{i\omega t}$. We have the potentials

$$\Phi(t, \vec{r}) = \iiint \frac{\rho(\vec{r}', t_{\text{ret}})}{|\vec{r} - \vec{r}'|} \mathrm{d}^3 r', \quad \vec{A}(t, \vec{r}) = \frac{1}{c} \iiint \frac{\vec{J}(\vec{r}', t_r)}{|\vec{r} - \vec{r}'|} \mathrm{d}^3 r',$$

which in the far field region become

$$\Phi(t, \vec{r} \gg \vec{r}') = \frac{1}{r} \iiint \rho(\vec{r}', t_{\text{ret}}) \mathrm{d}^3 r', \quad \vec{A}(t, \vec{r} \gg \vec{r}') = \frac{1}{cr} \iiint \vec{J}(\vec{r}', t_r) \mathrm{d}^3 r'.$$

We use the forms of the sources to obtain

$$\begin{aligned}
\Phi(t, \vec{r} \gg \vec{r}') &= \frac{1}{r} \iiint \rho_\omega(\vec{r}') e^{i\omega t_{\text{ret}}} \mathrm{d}^3 r' \\
&= \frac{1}{r} \iiint \rho_\omega(\vec{r}') \exp\left(i\omega\left(t - \frac{1}{c}|\vec{r} - \vec{r}'|\right)\right) \mathrm{d}^3 r' \\
&= \frac{1}{r} \iiint \rho_\omega(\vec{r}') \exp\left(i\omega\left(t - \frac{1}{c}\sqrt{r^2 + r'^2 - 2\vec{r} \cdot \vec{r}'}\right)\right) \mathrm{d}^3 r' \\
&\approx \frac{1}{r} \iiint \rho_\omega(\vec{r}') \exp\left(i\omega\left(t - \frac{r}{c}\sqrt{1 - 2\frac{\hat{\mathbf{r}} \cdot \vec{r}'}{r}}\right)\right) \mathrm{d}^3 r' \\
&\approx \frac{1}{r} \iiint \rho_\omega(\vec{r}') \exp\left(i\omega\left(t - \frac{r}{c}\left(1 - \frac{\hat{\mathbf{r}} \cdot \vec{r}'}{r}\right)\right)\right) \mathrm{d}^3 r',
\end{aligned}$$

and

$$\vec{A}(t, \vec{r} \gg \vec{r}') = \frac{1}{cr} \iiint \vec{J}_\omega(\vec{r}') \exp\left(i\omega\left(t - \frac{r}{c}\left(1 - \frac{\hat{\mathbf{r}} \cdot \vec{r}'}{r}\right)\right)\right) \mathrm{d}^3 r'.$$

The magnetic field is

$$\begin{aligned}
\vec{B} = \vec{\nabla}_r \times \vec{A} = \frac{1}{c} \vec{\nabla}_r \left(\frac{e^{i\omega(t - r/c)}}{r}\right) \times \iiint \vec{J}_\omega(\vec{r}') \exp\left(i\omega\frac{\hat{\mathbf{r}} \cdot \vec{r}'}{c}\right) \mathrm{d}^3 r' \\
+ \frac{e^{i\omega(t - r/c)}}{cr} \iiint \vec{\nabla}_r \left[\exp\left(i\omega\frac{\hat{\mathbf{r}} \cdot \vec{r}'}{c}\right)\right] \times \vec{J}_\omega(\vec{r}') \mathrm{d}^3 r',
\end{aligned}$$

where we used

$$\vec{\nabla}\times(\vec{a}f)=\epsilon_{ijk}\frac{\partial}{\partial x_j}(a_k f)=\epsilon_{ijk}\left(\frac{\partial f}{\partial x_j}\right)a_k+f\epsilon_{ijk}\frac{\partial}{\partial x_j}a_k=\left(\vec{\nabla}f\right)\times\vec{a}+f\left(\vec{\nabla}\times a\right).$$

We compute the first term and find

$$\begin{aligned}\vec{\nabla}_r\left(\frac{e^{i\omega(t-r/c)}}{r}\right)&=\vec{\nabla}_r\left(\frac{1}{r}\right)e^{i\omega(t-r/c)}+\frac{1}{r}\vec{\nabla}_r\left(e^{i\omega(t-r/c)}\right)\\&=-\left[\frac{1}{r^2}e^{i\omega(t-r/c)}+\frac{i\omega}{cr}e^{i\omega(t-r/c)}\right]\hat{\mathbf{r}}\\\left\{\begin{array}{c}r\gg c/\omega\\\implies\frac{1}{r^2}\ll\frac{\omega}{cr}\end{array}\right\}&\approx-\frac{i\omega}{cr}e^{i\omega(t-r/c)}\hat{\mathbf{r}},\end{aligned}$$

and the second term yields

$$\vec{\nabla}_r\left[\exp\left(i\omega\frac{\hat{\mathbf{r}}\cdot\vec{r}'}{c}\right)\right]=\vec{\nabla}_r\left[\exp\left(i\omega\frac{\vec{r}\cdot\vec{r}'}{cr}\right)\right]=0,$$

where we used

$$\begin{aligned}\vec{\nabla}\left(\vec{a}\cdot\vec{b}\right)&=\left(\vec{\nabla}\vec{a}\right)\cdot\vec{b}+\left(\vec{\nabla}\vec{b}\right)\cdot\vec{a}\\\implies\vec{\nabla}_r\left(\frac{\vec{r}\cdot\vec{r}'}{r}\right)&=\left(\hat{\mathbf{r}}\frac{\partial}{\partial r}\frac{\vec{r}}{r}\right)\cdot\vec{r}'=\left(\hat{\mathbf{r}}\hat{\mathbf{r}}\frac{\partial}{\partial r}\frac{r}{r}\right)\cdot\vec{r}'=0.\end{aligned}$$

We thus find the approximate magnetic field

$$\vec{B}\approx-\frac{i\omega}{c^2r}e^{i\omega(t-r/c)}\iiint\left(\hat{\mathbf{r}}\times\vec{J}_\omega\left(\vec{r}'\right)\right)\exp\left(i\omega\frac{\hat{\mathbf{r}}\cdot\vec{r}'}{c}\right)\mathrm{d}^3r'.$$

Similarly, the electric field is

$$\vec{E}\approx\frac{i\omega}{rc^2}e^{i\omega(t-r/c)}\iiint\left(\rho_\omega\hat{\mathbf{r}}-\vec{J}_\omega\left(\vec{r}'\right)\right)\exp\left(i\omega\frac{\hat{\mathbf{r}}\cdot\vec{r}'}{c}\right)\mathrm{d}^3r'.$$